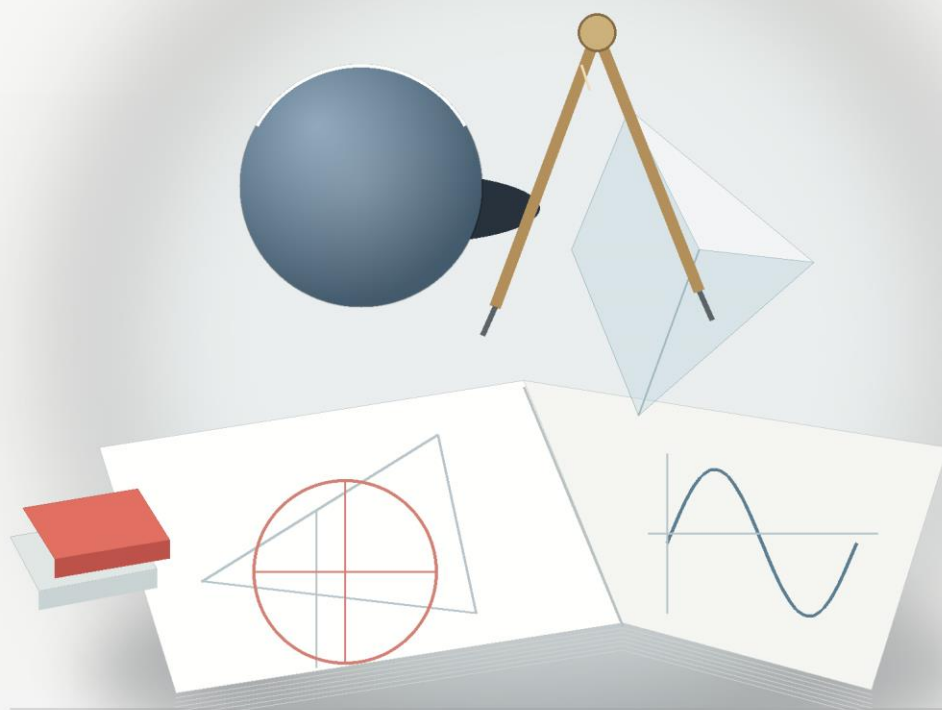


CURRENT TREND AND STUDIES IN **MATHEMATICS**



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Timelike Rotational Surface in Lorentzian Warped Product Manifold

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ABSTRACT

The concept of warped products, introduced by Bishop and O'Neill in 1969, provides a natural generalization of Riemannian product manifolds and has become an important tool in differential geometry and mathematical physics. Let B_1 and B_2 be two Riemannian manifolds with Riemannian metrics g_{B_1} and g_{B_2} , respectively. Let f be a smooth positive function on B_1 and also the product manifold of B_1 and B_2 denote by $B_1 \times B_2$. The warped product of B_1 and B_2 with warping function f is the Riemannian manifold $B_1 \times_f B_2 = (B_1 \times B_2, g_{B_1} + f^2 g_{B_2})$. Owing to their rich geometric structure, warped product spaces furnish a broad class of examples of Riemannian and semi-Riemannian manifolds and play a fundamental role in the construction of models arising in general relativity.

Especially, let f be a positive function on an open interval I and consider the warped product of $-I$ and the Euclidean 2-plane $M = E^2$ with warping function f , so we get

$$(\tilde{M}, \tilde{g}) = -I \times_f E^2 = (-I \times_f E^2, -dt^2 + f^2(t)(dx_1^2 + dx_2^2))$$

where t is the coordinate on I and x_1 and x_2 are coordinates on M . In this study, firstly we obtained the intrinsic and extrinsic curvature invariants of timelike rotational M in $\tilde{M} = -I \times_f E^2$.

Keywords – Warped product manifold, Lorentzian manifold, Rotational surface, Gauss curvature, Exterior Gauss curvature, Mean curvature.

INTRODUCTION

The study of timelike surfaces occupies a significant position in differential geometry, particularly within the context of Lorentzian and Minkowski spaces. Particularly, timelike rotational surfaces characterized by their invariance under a one-parameter group of isometries generating rotations, provide rich geometric structures such as, curvature properties, causal character, and the ambient space's metric signature. A foundational work in this subject is presented by Ganchev and Milousheva establishing a fundamental existence and uniqueness theorem, the study demonstrates that a zero-mean-curvature timelike surface (Ganchev & Milousheva, 2013). Their generalization to curved ambient spaces, particularly product spaces formed by the Cartesian product of two manifolds, introduces new challenges and intriguing geometric behaviors. One fundamental aspect in the study of rotational surfaces in product spaces is the characterization of their curvature properties, notably mean curvature and Gaussian curvature. These curvatures

not only encode intrinsic geometric information but also relate to physical models, such as minimal surfaces representing equilibrium states in variational problems. For instance, Güler and Kiş (2017) investigate rotational hypersurfaces in the product space $S^3(r) \times \mathbb{R}$, where $S^3(r)$ denotes the 3-sphere of radius r , embedded in five-dimensional Euclidean space E^5 . In that work, authors obtained explicit calculations of mean and Gaussian curvatures for these hypersurfaces, providing foundational results. An additional aspect of the research focuses on minimal rotational surfaces within product spaces that integrate constant curvature manifolds with circles or spheres. Gürpınar Arsan and Dursun (2018) investigate minimal rotational surfaces in the product spaces $Q_\varepsilon^2 \times S^1$, contingent on the parameter ε . Their analysis differentiates among various categories of rotational surfaces by examining the induced inner product along the rotational axis and identifies profile curves that define all minimal rotational surfaces in these contexts (Gürpınar Arsan & Dursun, 2018). This classification enhances the framework of minimal surface theory by integrating the concept of rotational symmetry within curved product spaces. These product spaces often possess additional structure, such as warped product metrics or Lorentzian signatures, which influence the curvature properties and stability of embedded surfaces.

The concept of warped products, introduced by Bishop and O'Neill in 1969 (Bishop & O'Neill, 1969), has become an important tool in differential geometry and mathematical physics. Owing to their rich geometric structure, warped product spaces furnish a broad class of examples of Riemannian and semi-Riemannian manifolds and play a fundamental role in the construction of models arising in general relativity. Studies of surfaces in warped product manifolds $I \times_f M$ have been investigated by several researchers, see (Chen, 2002; Alias and Dajczer, 2007a; Alias and Dajczer, 2007b; , Alias and Dajczer, 2008; Dillen et al., 2011; Chen, 2013; Aledo and Rubio, 2017; Dursun, 2022) and the references therein. In recent paper, Dursun and Turgay (2021) obtained a classification of all space-like and time-like constant angle surfaces in warped product space, $-I \times_f E_1^2$. In that paper, we will focus on how to obtain the curvatures of timelike rotational surface, including the Gauss curvature, exterior Gauss curvature and mean curvature. By employing the Levi-Civita connection of the ambient warped product

manifold and the corresponding Gauss and Weingarten formulas, explicit expressions for these curvature functions are obtained.

RESULTS AND DISCUSSION

In this section, firstly we provide some basic notations, formulas, definitions and results (O'Neill, 1983; Chen, 2017).

2.1. Basic concepts and formulas

Let I be an open interval in $\mathbb{R}$ and $f : I \rightarrow \mathbb{R}$ be a smooth strictly positive function, then, $\tilde{M} = -I \times_f E^2$ is a 3-dimensional Lorentzian warped product manifold whose metric tensor $\tilde{g}$ is given by

$$\tilde{g} = -dt^2 + f^2(t)(dx_1^2 + dx_2^2). \quad (1)$$

The canonical projections are denoted by $\Pi^1 : I \times \mathbb{R}^2 \rightarrow I$, $\Pi^2 : I \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Let V be the vector field in $\tilde{M}$, so we could decompose this vector as

$$V = V_0 \partial_t + \bar{V}. \quad (2)$$

Note that here, ∂_t is the comoving observer field in general relativity, V_0 is a function and $\bar{V}$ is a vector field satisfying $V_0 \stackrel{1}{=} \tilde{g}(V, \partial_t)$ and $\Pi^*(\bar{V}) = 0$.

Now we would like to give the following lemma directly obtained from [(Chen and Wei, 2009) in Lemma 2.1].

Lemma 2.1.1. The Levi-Civita connection $\tilde{\nabla}$ of $\tilde{M}$

$$\tilde{\nabla}_U V = D_U V + \frac{f}{f'}(U_0 \bar{V} + V_0 \bar{U} + \tilde{g}(\bar{U}, \bar{V}) \partial_t) \quad (3)$$

whenever the tangent vectors U, V of $\tilde{M} = -I \times_f E^2$, where D denotes the connection of E^2 .

Let $\sigma : M \rightarrow -I \times_f E^2$ be an isometric immersion of M in $-I \times_f E^2$, and let g be the pulled-back metric of $\tilde{g}$ on M , ξ be a unit normal vector field on M , and denote by S the associated shape operator. The equations of Gauss and Weingarten are given by

$$\tilde{\nabla}_U V = \nabla_U V + \text{II}(U, V), \quad (4)$$

$$\tilde{\nabla}_U \xi = -SU, \quad (5)$$

where U, V are tangent and ξ is normal to M , respectively. Also ∇ is the induced metric connection, II is the second fundamental form and S is the shape operator of M . Note that S and II are related by

$$\tilde{g}(\Pi(U, V), \xi) = g(SU, V), \quad (6)$$

$$\Pi(U, V) = g(SU, V)\xi. \quad (7)$$

Moreover, The Gauss and Codazzi equations are given, respectively, by

$$R(U, V)Z, W) = \langle \Pi(V, Z), \Pi(U, W) \rangle - \langle \Pi(U, Z), \Pi(V, W) \rangle, \quad (8)$$

$$(\tilde{\nabla}_U \Pi)(V, Z) = (\tilde{\nabla}_V \Pi)(U, Z), \quad (9)$$

where R is the curvature tensor of M and $\tilde{\nabla}h$ is defined by

$$(\tilde{\nabla}_U \Pi)(V, Z) = \nabla^\perp \Pi(V, Z) - \Pi(\nabla_U V, Z) - \Pi(V, \nabla_U Z). \quad (10)$$

The Gaussian curvature K , the exterior Gauss curvature K_{ext} and the mean curvature vector field H of M in $\tilde{M}$ is defined by, respectively,

$$K = \frac{\langle R(U, V)V, U \rangle}{\langle U, U \rangle \langle V, V \rangle - \langle U, V \rangle^2} \quad (11)$$

$$K_{\text{ext}} = \det S, \quad (12)$$

$$H = \frac{1}{2} \text{tr} \Pi \quad (13)$$

where $\{U, V\}$ is a frame field for the tangent bundle of M . As well known that M is said to be flat, external flat or minimal if K , K_{ext} or H vanishes identically on M .

Let $U = (U_1, U_2, U_3)$, $V = (V_1, V_2, V_3)$ be two vectors in $T\tilde{M}$. Then the warped vector product of these vectors is calculated by

$$U \times_f V = (-f^2(t)(U_2 V_3 - V_2 U_3), (U_3 V_1 - V_3 U_1), (U_1 V_2 - V_1 U_2)).$$

2.2. Rotational Surfaces in Warped Product Manifold

In this subsection, firstly we will give the definition of the rotational surface M in the warped product manifold $\tilde{M} = -I \times_f E^2$. Then, we will obtain the curvatures of the surface M .

Definition 2.2.1. The rotation about the t -axis given by

$$R_\vartheta : \tilde{M} \rightarrow \tilde{M} : (t, x_1, x_2) \rightarrow (t, x_1 \cos \vartheta - x_2 \sin \vartheta, x_1 \sin \vartheta + x_2 \cos \vartheta) \quad (14)$$

is a one-parameter group of isometries which left the t -axis pointwise invariant. Let $\alpha(u) = (u, y(u), 0)$, $u \in J$, be a smooth curve in $-I \times_f E^2$ defined on an open interval J . So, the rotational surface about the t -axis in $-I \times_f E^2$ with a profile curve is defined by $\sigma(u, v) = R(\alpha(u))$. We can assume that the profile curve is lying in the spacelike plane of $\tilde{M}$. Then, the timelike rotational surface M is parametrized by,

$$\sigma(u, v) = (u, y(u) \cos v, y(u) \sin v). \quad (15)$$

Now, let M be a rotational surface in $\tilde{M} = -I \times_f E^2$ defined as in (15). Then, there is an orthonormal frame field $\{e_1, e_2; \xi\}$ of M obtained as

$$e_1 = \frac{1}{\sqrt{1-f^2 y'^2}} (1, y' \cos v, y' \sin v), \quad (16)$$

$$e_2 = \left(0, \frac{-\sin v}{f}, \frac{\cos v}{f}\right), \quad (17)$$

$$\xi = \frac{f}{\sqrt{1-f^2 y'^2}} \left(y', \frac{\cos v}{f^2}, \frac{\sin v}{f^2}\right) \quad (18)$$

with $\tilde{g}(e_2, e_2) = \tilde{g}(\xi, \xi) = 1$ and $\tilde{g}(e_1, e_1) = -1$ where the metric tensor $\tilde{g}$ is defined as in (1). Since ξ is spacelike, we could say that the surface M is timelike. Moreover, here and throught the paper the functions f and y depend on the parameter $u \in I$.

Proposition 2.2.1. Let M be a timelike rotational surface in $\tilde{M} = -I \times_f E^2$. Then, there is an orthonormal frame field $\{e_1, e_2; \xi\}$ with $\tilde{g}(\xi, \xi) = \tilde{g}(e_1, e_1) = -1$ such that the shape operator S is diagonalizable, i.e.,

$$S = \begin{pmatrix} \frac{f'y'(2-f^2 y'^2) + fy''}{(1-f^2 y'^2)^{3/2}} & 0 \\ 0 & \frac{-1 - yy'ff'}{fy\sqrt{1-f^2 y'^2}} \end{pmatrix} \quad (19)$$

and the Levi-Civita connection ∇ of M is given by

$$\begin{aligned} \nabla_{e_1} e_1 &= \nabla_{e_1} e_2 = 0, \\ \nabla_{e_2} e_1 &= \frac{fy' + yf'}{fy\sqrt{1-f^2 y'^2}} e_2, \\ \nabla_{e_2} e_2 &= \frac{fy' + yf'}{fy\sqrt{1-f^2 y'^2}} e_1. \end{aligned} \quad (20)$$

Proof. Let M be a timelike rotational surface in $\tilde{M}$. So, we have the tangent vectors $\{e_1, e_2\}$ given in (16) and (17). By considering these in (2) and then in (3), we get the Levi-Civita connections $\tilde{\nabla}$ of $\tilde{M}$ as the following

$$\begin{aligned}\tilde{\nabla}_{e_1} e_1 &= \left(0, 0, \frac{fy'(2-f^2y'^2)+fy''}{(1-f^2y'^2)^{3/2}} \right), \\ \tilde{\nabla}_{e_1} e_2 &= (0, 0, 0),\end{aligned}\tag{21}$$

$$\tilde{\nabla}_{e_2} e_1 = \left(0, -\frac{fy'+yf}{fy\sqrt{1-f^2y'^2}}, 0 \right),$$

$$\tilde{\nabla}_{e_2} e_2 = \left(-\frac{fy'+yf}{fy\sqrt{1-f^2y'^2}}, 0, -\frac{1+yy'ff}{fy\sqrt{1-f^2y'^2}} \right).$$

So, if we separate the tangent and normal parts of these connections, we have directly the Levi-Civita connection ∇ of M as in (20) from the tangent parts. Moreover, from the normal parts we will get the second fundamental form coefficients, $II(e_i, e_i)$, $i=1,2$. So by considering these and (18) in (6), we get the shape operator S as in (19).

Theorem 2.2.2 Let M be a timelike rotational surface in $\tilde{M}$. Then, the exterior Gauss curvature K_{ext} , the mean curvature vector field H and the Gaussian curvature K of M in $\tilde{M}$ is obtained by, respectively,

$$K_{\text{ext}} = \frac{(1+yy'ff)(fy'(2-f^2y'^2)+fy'')}{fy(1-f^2y'^2)^2}\tag{22}$$

$$H = \frac{1+3yy'ff-2yf f^3 y'^3 + f^2(-y'^2+yy'')}{fy(1-f^2y'^2)^{3/2}}\tag{23}$$

$$K = \frac{y'^3 f'^2 f' - y'' f - y' f' (2+yy''f^2) - y f'' - y f y'^2 (f'^2 - ff'')}{yf(1-f^2y'^2)^2}\tag{24}$$

Proof. Let M be a timelike rotational surface given in (15). Then, from (12) and (19) and also (13) and (19), we get directly the exterior Gauss curvature (22) and the mean curvature vector field (23), respectively. On the other hand, by considering (19) into (8) and (11), it follows that the Gaussian curvature can be obtained as in (24).

2.3. Results

As well known that M is said to be flat, external flat or minimal if K , K_{ext} or H vanishes identically on M . In this section, we investigate these conditions.

2.3.1. External Flat Rotational Surfaces

Consider the timelike rotational surface M parametrized in (15). From Theorem 2.2.2., its the exterior Gauss curvature K_{ext} is obtained by (22), hence such a surface is external flat if and only if

$$(1 + yf'y') \left(y' (2 - f^2 y'^2) f' + f y'' \right) = 0. \quad (25)$$

By solving the equation (25), we get

$$y(u) = \pm \sqrt{2} \sqrt{c_1 - \int_1^u \frac{ds}{f(s)f'(s)}} \quad (26)$$

or

$$y(u) = \pm c_2 - \int_1^u \frac{ds}{\sqrt{f(s)^2 + c_1 f(s)^4}} \quad (27)$$

where $c_1, c_2 \in \mathbb{R}$. So we obtain the parametrization of the timelike rotational surface M being external flat in $\tilde{M} = -I \times_f E^2$ as,

$$\sigma(u, v) = \left(u, \pm \sqrt{2} \sqrt{c_1 - \int_1^u \frac{ds}{f(s)f'(s)}} \cos v, \pm \sqrt{2} \sqrt{c_1 - \int_1^u \frac{ds}{f(s)f'(s)}} \sin v \right) \quad (28)$$

or

$$\left(u, \pm c_2 - \int_1^u \frac{ds}{\sqrt{f[s]^2 + c_1 f[s]^4}} \cos v, \pm c_2 - \int_1^u \frac{ds}{\sqrt{f[s]^2 + c_1 f[s]^4}} \sin v \right) \quad (29)$$

2.3.2. Minimal Rotational Surfaces

Consider the timelike rotational surface M parametrized in (15). From Theorem 2.2.2., its the mean curvature H is obtained by (24), hence such a surface is minimal if and only if

$$1 + 3yy'ff'' - 2yf f'^3 y'^3 + f^2 (-y'^2 + yy'') = 0. \quad (30)$$

2.3.3. Flat Rotational Surfaces

Consider the Lorentzian rotational surface M parametrized in (15). From Theorem 2.2.2., its the Gauss curvature K is obtained by (22), hence such a surface is flat if and only if

$$y'^3 f'^2 - y'' f' - y' f' (2 + y y'' f'^2) - y f'' - y f y'^2 (f'^2 - f f'') = 0. \quad (31)$$

2.4. Discussion

In that paper, we will focus on how to obtain the curvatures of timelike rotational surface in warped product spaces, including the intrinsic and extrinsic curvature invariants. By employing the Levi-Civita connection of the ambient warped product manifold and the corresponding Gauss and Weingarten formulas, explicit expressions for these curvature functions are obtained. The obtained results provide a better understanding of the interplay between the geometry of the ambient warped product space and the intrinsic geometry of the surface of revolution.

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Spacelike Surface of Revolution in $-I \times_f E^2$

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ABSTRACT

The concept of warped products, introduced by Bishop and O'Neill in 1969, provides a natural generalization of Riemannian product manifolds and has become an important tool in differential geometry and mathematical physics. Let F_1 and F_2 be two Riemannian manifolds with Riemannian metrics g_{F_1} and g_{F_2} , respectively. Let f be a smooth positive function on F_1 and also the product manifold of F_1 and F_2 denote by $F_1 \times F_2$. The warped product of F_1 and F_2 with warping function f is the Riemannian manifold $F_1 \times_f F_2 = (F_1 \times F_2, g_{F_1} + f^2 g_{F_2})$. Owing to their rich geometric structure, warped product spaces furnish a broad class of examples of Riemannian and semi-Riemannian manifolds and play a fundamental role in the construction of models arising in general relativity.

Especially, let f be a positive function on an open interval I and consider the warped product of $-I$ and the Euclidean 2-plane $M = E^2$ with warping function f , so we get

$$(\tilde{M}, \tilde{g}) = -I \times_f E^2 = (-I \times_f E^2, -dt^2 + f^2(t)(dx_1^2 + dx_2^2))$$

where t is the coordinate on I and x_1 and x_2 are coordinates on M . In this study, firstly we obtained the intrinsic and extrinsic curvature invariants of spacelike surface of revolution M in $\tilde{M} = -I \times_f E^2$.

Keywords – Warped product manifold, Surface of revolution, Intrinsic curvature, Extrinsic curvature.

INTRODUCTION

The study of surfaces of revolution holds a central position in differential geometry and mathematical physics, largely due to their rich geometric properties and wide-ranging applications across various disciplines. Formed by rotating a planar curve around a specific axis, these surfaces inherently exhibit strong rotational symmetry. In the literature, surfaces of revolution have been extensively examined within the context of constant mean curvature (CMC) surfaces, minimal surfaces, and their generalizations across various spaces. Current research reflects a multifaceted investigation, spanning from classical geometric frameworks to modern computational and physical realizations.

The fundamental theory of surfaces of revolution with constant mean curvature was established by C. Delaunay in the 19th century. By solving the relevant differential equations and developing geometric constructions based on the evolutes of the generating curves, Delaunay (1841) completely

classified all constant mean curvature (CMC) surfaces of revolution in Euclidean space. Later research expanded far beyond the Euclidean framework to include Lorentz and non-Euclidean geometries. The differential-geometric properties and classification problems of these surfaces within various settings have been extensively studied in the literature (Arslan et al., 2012; Choi et al., 2013; Kelleci Akbay, 2021; Al-Zoubi et al., 2022; da Silva and Lopez, 2026).

Generalizing these to curved ambient spaces—particularly product spaces formed by the Cartesian product of two manifolds—brings with it new challenges and intriguing geometric behaviors. A fundamental aspect in the study of surfaces of revolution in product spaces is characterizing their curvature properties, primarily mean curvature and Gaussian curvature. These curvatures not only contain intrinsic geometric information but are also related to physical models, such as minimal surfaces that represent equilibrium states in variational problems. For instance, Güler and Kişı (2017) investigate rotational hypersurfaces in the product space $S^3(r) \times \mathbb{R}$, where $S^3(r)$ denotes the 3-sphere of radius r , embedded in five-dimensional Euclidean space E^5 . In this study, the authors provide explicit calculations of mean and Gaussian curvatures for the relevant hypersurfaces and present fundamental results that facilitate further geometric analysis. This approach illustrates how classical concepts of curvature can be naturally extended to higher-dimensional and more complex ambient spaces. Another aspect of the research focuses on minimal rotational surfaces in product spaces that combine manifolds of constant curvature with circles or spheres. Gürpınar Arsan and Dursun (2018) investigate minimal rotational surfaces in product spaces of the form $Q_{\epsilon^2} \times S^1$, parameterized by ϵ . Their work distinguishes between various categories of rotational surfaces by considering the inner product reduced along the axis of rotation and identifies the profile curves that characterize all minimal rotational surfaces in these contexts (Gürpınar Arsan & Dursun, 2018). This classification enriches the framework of minimal surface theory by integrating the concept of rotational symmetry into the setting of product spaces. These product spaces often possess additional structures—such as warped product metrics or Lorentzian signatures—that influence the curvature properties and stability of surfaces embedded within them.

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RESULTS AND DISCUSSION

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$$\tilde{g} = -dt^2 + f^2(t)(dx_1^2 + dx_2^2). \quad (1)$$

The canonical projections are denoted by $\Pi^1 : I \times \mathbb{R}^2 \rightarrow I$, $\Pi^2 : I \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Let V be the vector field in $\tilde{M}$, so we could decompose this vector as

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Note that here, ∂_t is the comoving observer field in general relativity, V_0 is a function and $\bar{V}$ is a vector field satisfying $V_0 \stackrel{1}{=} \tilde{g}(V, \partial_t)$ and $\Pi^*(\bar{V}) = 0$.

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Lemma 2.1.1. The Levi-Civita connection $\tilde{\nabla}$ of $\tilde{M}$

$$\tilde{\nabla}_U V = D_U V + \frac{f}{f} (U_0 \bar{V} + V_0 \bar{U} + \tilde{g}(\bar{U}, \bar{V}) \partial_t) \quad (3)$$

whenever the tangent vectors U, V of $\tilde{M} = -I \times_f E^2$, where D denotes the connection of E^2 .

Let $\sigma : M \rightarrow -I \times_f E^2$ be an isometric immersion of M in $-I \times_f E^2$, and let g be the pulled-back metric of $\tilde{g}$ on M , N be a unit normal vector field on M , and denote by S the associated shape operator. The equations of Gauss and Weingarten are given by

$$\tilde{\nabla}_U V = \nabla_U V + II(U, V), \quad (4)$$

$$\tilde{\nabla}_U N = -SU, \quad (5)$$

where U, V are tangent and ξ is normal to M , respectively. Also ∇ is the induced metric connection, II is the second fundamental form and S is the shape operator of M . Note that S and II are related by

$$\tilde{g}(II(U, V), N) = g(SU, V), \quad (6)$$

$$II(U, V) = g(SU, V)N. \quad (7)$$

Moreover, The Gauss and Codazzi equations are given, respectively, by

$$\langle R(U, V)Z, W \rangle = \langle II(V, Z), II(U, W) \rangle - \langle II(U, Z), II(V, W) \rangle, \quad (8)$$

$$(\tilde{\nabla}_U II)(V, Z) = (\tilde{\nabla}_V II)(U, Z), \quad (9)$$

where R is the curvature tensor of M and $\tilde{\nabla}h$ is defined by

$$(\tilde{\nabla}_U II)(V, Z) = \nabla^\perp II(V, Z) - II(\nabla_U V, Z) - II(V, \nabla_U Z). \quad (10)$$

The Gaussian curvature K , the exterior Gauss curvature K_{ext} and the mean curvature vector field H of M in $\tilde{M}$ is defined by, respectively,

$$K = \frac{\langle R(U, V)V, U \rangle}{\langle U, U \rangle \langle V, V \rangle - \langle U, V \rangle^2} \quad (11)$$

$$K_{\text{ext}} = \det S, \quad (12)$$

$$H = \frac{1}{2} \text{tr } II \quad (13)$$

where $\{U, V\}$ is a frame field for the tangent bundle of M . As well known that M is said to be flat, external flat or minimal if K , K_{ext} or H vanishes identically on M .

Let $U = (u_1, u_2, u_3)$, $V = (v_1, v_2, v_3)$ be two vectors in $T\tilde{M}$. Then the warped vector product of these vectors is calculated by

$$U \times_f V = (-f^2(t)(u_2 v_3 - v_2 u_3), (u_3 v_1 - v_3 u_1), (u_1 v_2 - v_1 u_2)).$$

2.2. Surfaces of Revolution in Warped Product Manifold

In this subsection, firstly we will give the definition of the surface of revolution M in the warped product manifold $\tilde{M} = -I \times_f E^2$. Then, we will obtain the curvatures of the surface M .

Definition 2.2.1. The rotation about the t -axis given by

$$R_\vartheta : \tilde{M} \rightarrow \tilde{M} : (t, u_1, u_2) \rightarrow (t, u_1 \cos \vartheta - u_2 \sin \vartheta, u_1 \sin \vartheta + u_2 \cos \vartheta) \quad (14)$$

is a one-parameter group of isometries which left the t -axis pointwise invariant. Let $\beta(u) = (u, y(u), 0)$, $u \in J$, be a smooth curve in $-I \times_f E^2$ defined on an open interval J . So, the rotational surface about the t -axis in $-I \times_f E^2$ with a profile curve is defined by $\sigma(u, v) = R(\beta(u))$. We can assume that the profile curve is lying in the spacelike plane of $\tilde{M}$. Then, the Lorentzian rotational surface M is parametrized by,

$$\sigma(u, v) = (u, y(u) \cos v, y(u) \sin v). \quad (15)$$

Now, let M be a rotational surface in $\tilde{M} = I \times_f E_1^2$ defined as in (15). Then, there is an orthonormal frame field $\{e_1, e_2; N\}$ of M obtained as

$$e_1 = \left(\frac{1}{\sqrt{-1+f^2 y'^2}}, \frac{y' \cos v}{\sqrt{-1+f^2 y'^2}}, \frac{y' \sin v}{\sqrt{-1+f^2 y'^2}} \right), \quad (16)$$

$$e_2 = \left(0, \frac{-\sin v}{f}, \frac{\cos v}{f} \right), \quad (17)$$

$$N = \left(\frac{-f y'}{\sqrt{-1+f^2 y'^2}}, \frac{\cos v}{f \sqrt{-1+f^2 y'^2}}, \frac{\sin v}{f \sqrt{-1+f^2 y'^2}} \right) \quad (18)$$

with $\tilde{g}(e_2, e_2) = 1$ and $\tilde{g}(N, N) = \tilde{g}(e_1, e_1) = -1$ where the metric tensor $\tilde{g}$ is defined as in (1). Since N is timelike, we could say that the surface M is spacelike. Moreover, here and through the paper the functions f and y depend on the parameter $u \in I$.

Proposition 2.2.1. Let M be a spacelike surface of revolution in $\tilde{M} = -I \times_f E^2$. Then, there is an orthonormal frame field $\{e_1, e_2; N\}$ with $\tilde{g}(N, N) = \tilde{g}(e_1, e_1) = -1$ such that the shape operator S is diagonalizable, i.e.,

$$S = \begin{pmatrix} \frac{f'y'(2-f^2y'^2)+fy''}{(-1+f^2y'^2)^{3/2}} & 0 \\ 0 & \frac{-1-yy'ff'}{fy\sqrt{-1+f^2y'^2}} \end{pmatrix} \quad (19)$$

and the Levi-Civita connection ∇ of M is given by

$$\begin{aligned} \nabla_{e_1} e_1 &= \nabla_{e_1} e_2 = 0, \\ \nabla_{e_2} e_1 &= \frac{fy' + yf'}{fy\sqrt{-1+f^2y'^2}} e_2, \\ \nabla_{e_2} e_2 &= \frac{1 + yy'ff'}{fy\sqrt{-1+f^2y'^2}} e_1. \end{aligned} \quad (20)$$

Proof. Let M be a spacelike rotational surface in $\tilde{M}$. So, we have the tangent vectors $\{e_1, e_2\}$ given in (16) and (17). By considering these in (2) and then in (3), we get the Levi-Civita connections $\tilde{\nabla}$ of $\tilde{M}$ as the following

$$\begin{aligned} \tilde{\nabla}_{e_1} e_1 &= \left(0, 0, \frac{fy'' + f'y'(2-f^2y'^2)}{(-1+f^2y'^2)^{3/2}} \right), \\ \tilde{\nabla}_{e_1} e_2 &= (0, 0, 0), \\ \tilde{\nabla}_{e_2} e_1 &= \left(0, \frac{fy' + yf'}{fy\sqrt{-1+f^2y'^2}}, 0 \right), \\ \tilde{\nabla}_{e_2} e_2 &= \left(-\frac{fy' + yf'}{fy\sqrt{-1+f^2y'^2}}, 0, -\frac{1 + yy'ff'}{fy\sqrt{-1+f^2y'^2}} \right). \end{aligned} \quad (21)$$

So, if we separate the tangent and normal parts of these connections, we have directly the Levi-Civita connection ∇ of M as in (20) from the tangent parts. Moreover, from the normal parts we will get the second fundemantal form coefficients, $II(e_i, e_i)$, $i=1,2$. So by considering these and (18) in (6), we get the shape operator S as in (19).

Theorem 2.2.2 Let M be a spacelike surface of revolution in $\tilde{M}$. Then, the exterior Gauss curvature K_{ext} , the mean curvature vector field H and the Gaussian curvature K of M in $\tilde{M}$ is obtained by, respectively,

$$K_{\text{ext}} = \frac{(1 + yy'ff')(f'y'(2 - f^2y'^2) + fy'')}{fy(-1 + f^2y'^2)^2} \quad (22)$$

$$H = \frac{1 + 3yy'ff' - 2yf'f^3y'^3 + f^2(-y'^2 + yy'')}{fy(-1 + f^2y'^2)^{3/2}} \quad (23)$$

$$K = \frac{-y'^3f^2f' + y''f + y'f'(2 + yy''f^2) + yf'' + yy'^2(ff'^2 - f^2f'')}{yf(-1 + f^2y'^2)^2} \quad (24)$$

Proof. Let M be a spacelike surface of revolution given in (15). Then, from (12) and (19) and also (13) and (19), we get directly the exterior Gauss curvature (22) and the mean curvature vector field (23), respectively. On the other hand, by considering (19) into (8) and (11), it follows that the Gaussian curvature can be obtained as in (24).

2.3. Results

As well known that M is said to be flat, external flat or minimal if K , K_{ext} or H vanishes identically on M . In this section, we investigate these conditions.

2.3.1. External Flat Surface of Revolution

Consider the spacelike rotational surface M parametrized in (15). From Theorem 2.2.2., its the exterior Gauss curvature K_{ext} is obtained by (22), hence such a surface is external flat if and only if

$$(1 + yfy'f')(f'y'(2 - f^2y'^2) + fy'') = 0. \quad (25)$$

By solving the equation (25), we get

$$y(u) = \pm\sqrt{2} \sqrt{c_1 - \int_1^u \frac{ds}{f(s)f'(s)}} \quad (26)$$

or

$$y(u) = \pm c_2 - \int_1^u \frac{ds}{\sqrt{f(s)^2 + c_1 f(s)^4}} \quad (27)$$

where $c_1, c_2 \in \mathbb{R}$. So we obtain the parametrization of the spacelike surface of revolution M being external flat in $\tilde{M} = -I \times_f E^2$ as,

$$\sigma(u, v) = \left(u, \pm \sqrt{2} \sqrt{c_1 \pm \int_1^u \frac{ds}{f(s)f'(s)}} \cos v, \pm \sqrt{2} \sqrt{c_1 \pm \int_1^u \frac{ds}{f(s)f'(s)}} \sin v \right) \quad (28)$$

or

$$\left(u, c_2 \pm \int_1^u \frac{ds}{\sqrt{f(s)^2 + c_1 f(s)^4}} \cos v, c_2 \pm \int_1^u \frac{ds}{\sqrt{f(s)^2 + c_1 f(s)^4}} \sin v \right) \quad (29)$$

2.3.2. Minimal Surface of Revolution

Consider the spacelike surface of revolution M parametrized in (15). From Theorem 2.2.2., its the mean curvature H is obtained by (23), hence such a surface is minimal if and only if

$$1 + 3yy'ff'' - 2yf'f^3y'^3 + f^2(-y'^2 + yy'') = 0. \quad (30)$$

2.3.3. Flat Surface of Revolution

Consider the spacelike surface of revolution M parametrized in (15). From Theorem 2.2.2., its the Gauss curvature K is obtained by (24), hence such a surface is flat if and only if

$$y'^3 f'^2 f'' - y'' f - y' f' f' (2 + yy'' f^2) - y f'' - y f y'^2 (f'^2 - ff'') = 0. \quad (31)$$

2.4. Discussion

In that paper, we will focus on how to obtain the curvatures of spacelike surface of revolution in warped product spaces, including the intrinsic and extrinsic curvature invariants. By employing the Levi-Civita connection of the ambient warped product manifold and the corresponding Gauss and Weingarten formulas, explicit expressions for these curvature functions are obtained. The obtained results provide a better understanding of the interplay between the geometry of the ambient warped product space and the intrinsic geometry of the surface of revolution. Furthermore, these findings contribute to broader studies on submanifold theory and can provide a foundation for future research on specific surface classes and their applications in geometric analysis and mathematical physics.

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An Alternative Approach to Associated Curves via Torsion Modified Orthogonal Frame in Euclidean Space

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ABSTRACT

This paper investigates the geometric properties of associated curves in three-dimensional Euclidean space by using a torsion-based modified orthogonal frame (TMOF). The associated curves in the tangent, principal normal, and binormal directions are constructed within this framework. For each construction, the corresponding moving frame, curvature, torsion, and distance relations are derived. The obtained formulas are then used to characterize several geometric cases, including planar involute-type curves and constant-distance partner curves related to Bertrand- and Mannheim-type constructions. These results show how the torsion function affects the geometry of the associated curves and determine the conditions under which special curve relations occur. Numerical examples based on a catenary are included to illustrate the formulas and the resulting curves.

Keywords – Associated curves, Torsion-modified orthogonal frame, Euclidean 3-space, Involute curves, Bertrand curves, Mannheim curves, Catenary curve

INTRODUCTION

The study of space curves and their associated partner curves is a fundamental topic in differential geometry, with extensive applications in kinematics, structural engineering, and robotics (O'Neill, 2016). A classical approach in curve theory involves constructing new auxiliary curves from a base curve to analyze their mutual geometric relationships (Choi and Kim, 2012; Qian and Kim, 2015; Kaya and Önder, 2017). Families of associated curves, including Bertrand pairs, Mannheim pairs, and involute-evolute configurations, have been broadly studied to characterize primary curve properties (Güven, 2020; Şahiner, 2019). Recently, Şenyurt et al. (2022) expanded this domain by introducing classes of associated curves whose position vectors systematically lie in the osculating, normal, and rectifying planes of partner curves.

Traditionally, the Frenet–Serret frame serves as the standard tool for analyzing spatial curves. However, the classical Frenet frame becomes undefined at inflection points where the curvature vanishes, which limits its direct applicability in such regions. While alternative frameworks, such as the Bishop (parallel transport) frame, handle inflection points continuously, their construction requires integral formulations to define the normal direction

vectors. To provide a computationally direct alternative, Sasai (1984) introduced the curvature-based Modified Orthogonal Frame (MOF). By scaling the principal normal and binormal vectors via the curvature function, the MOF ensures smooth vector fields along the curve while preserving classical geometric properties (Lone et al., 2019; Elsayied et al., 2021, 2022; Elsharkawy and Elshenhab, 2022; Saad et al., 2024; Damar et al., 2024, 2025; Baek and Yüksel, 2025).

In a similar spirit, Bükçü and Karacan (2016) proposed the Torsion-Modified Orthogonal Frame (TMOF) as a torsion-based alternative. Instead of scaling by curvature, the TMOF scales the principal normal and binormal vectors using the torsion function of the base curve. This construction allows one to express the moving frame and its structural equations directly through the torsion of the curve, without requiring auxiliary integral transforms. Consequently, the TMOF provides an alternative frame description that is particularly convenient when analyzing geometric configurations driven by torsion. In this setting, the analysis is systematically conducted on open intervals where the torsion function is non-vanishing ($\tau \neq 0$).

Motivated by these structural properties, the present paper investigates associated curves constructed within the TMOF framework in three-dimensional Euclidean space. We systematically define tangent, principal normal, and binormal associated curves by placing their position vectors in the primary planes of their partner curves. For each construction, we derive the moving frame relations, structural curvatures, torsions, and distance equations. These analytical formulations are then applied to characterize special geometric configurations, including planar involute-type curves and constant-distance partner curves of Bertrand and Mannheim types. Finally, to illustrate the theoretical results, a numerical example based on a spatial catenary curve is presented on intervals where its torsion remains non-zero.

MATERIALS AND METHODS

Let $\gamma: I \subseteq \mathbb{R} \rightarrow \mathbb{E}^3$ be a unit-speed regular space curve with non-zero curvature $\kappa(s)$ and non-zero torsion $\tau(s)$. In contrast to the classical Frenet frame, we introduce the Torsion-Modified Orthogonal Frame (TMOF), denoted by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$. In this frame, the principal normal and binormal vectors are scaled by the torsion function $\tau(s)$. The relations between the classical

Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ and the TMOF are given by $\mathbf{T} = \mathbf{t}$, $\mathbf{N} = \tau \mathbf{n}$, and $\mathbf{B} = \tau \mathbf{b}$.

The derivative formulas of the Torsion-Modified Orthogonal Frame with respect to the arc-length parameter s are governed by the following system of differential equations (Bükçü and Karacan, 2016):

$$\begin{aligned} \mathbf{T}' &= \frac{\kappa}{\tau} \mathbf{N}, \quad \mathbf{N}' = -\kappa \tau \mathbf{T} + \frac{\tau'}{\tau} \mathbf{N} + \tau \mathbf{B}, \quad \mathbf{B}' = -\tau \mathbf{N} \\ &\quad + \frac{\tau'}{\tau} \mathbf{B}. \end{aligned} \quad (1)$$

For an arbitrary curve $\alpha(t)$, the TMOF vectors and the curve parameters (κ, τ) can be explicitly calculated as follows:

$$\mathbf{T} = \frac{\alpha'}{\|\alpha'\|}, \quad \mathbf{B} = \tau \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}, \quad \mathbf{N} = \mathbf{B} \times \mathbf{T}, \quad (2)$$

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \quad \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \times \alpha''\|^2}. \quad (3)$$

RESULTS AND DISCUSSION

Tangent-Associated Curves in the TMOF Setting

To provide a clear geometric frame for associated curves within the Torsion-Modified Orthogonal Frame (TMOF), let $\gamma: I \rightarrow \mathbb{R}^3$ be a unit-speed regular base curve with curvature $\kappa(s) > 0$, torsion $\tau(s) \neq 0$, and TMOF $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$. An associated partner curve $\tilde{\gamma}$ is generated by displacing γ along one of its moving frame vectors $V \in \{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ by a sufficiently smooth distance function $\delta(s)$:

$$\tilde{\gamma}(s) = \gamma(s) + \delta(s)V(s). \quad (4)$$

Let $\{\tilde{\mathbf{T}}, \tilde{\mathbf{N}}, \tilde{\mathbf{B}}\}$ denote the TMOF of the partner curve $\tilde{\gamma}$, with strictly positive curvature $\tilde{\kappa}(s)$ and torsion $\tilde{\tau}(s)$. To establish the geometric correlation between γ and $\tilde{\gamma}$, we define three distinguished unit direction fields. These fields represent arbitrary fixed directions within the osculating, normal, and rectifying planes of the partner curve $\tilde{\gamma}$, respectively:

$$\tilde{\mathbf{O}}(s) = \frac{u(s)\tilde{\mathbf{T}}(s) + v(s)\tilde{\mathbf{N}}(s)}{\sqrt{u(s)^2 + (v(s)\tilde{\tau}(s))^2}}, \quad (5)$$

$$\tilde{\mathbf{P}}(s) = \frac{w(s)\tilde{\mathbf{N}}(s) + x(s)\tilde{\mathbf{B}}(s)}{\tilde{\tau}(s)\sqrt{w(s)^2 + x(s)^2}}, \quad (6)$$

$$\tilde{\mathbf{R}}(s) = \frac{y(s)\tilde{\mathbf{T}}(s) + z(s)\tilde{\mathbf{B}}(s)}{\sqrt{y(s)^2 + (z(s)\tilde{\tau}(s))^2}}, \quad (7)$$

where $u(s), v(s), w(s), x(s), y(s), z(s)$ are smooth real-valued functions in the case of $\mathbf{T}-\tilde{\mathbf{P}}$ partner curves, whereas they are taken as constant real numbers for the other types of partner curves.

In the present section, we specifically focus on the tangential displacement case ($V = \mathbf{T}$).

Definition 1. Let γ and $\tilde{\gamma}$ be regular space curves defined with the Torsion-Modified Orthogonal Frame (TMOF). If the tangent vector $\mathbf{T}(s)$ of the curve γ and the osculating-plane vector $\tilde{\mathbf{O}}(s)$ of the curve $\tilde{\gamma}$ are collinear, i.e., $\mathbf{T}(s) = \tilde{\mathbf{O}}(s)$, then $\tilde{\gamma}$ is called a $\mathbf{T}-\tilde{\mathbf{O}}$ associated curve of γ .

Theorem 1. Let γ be a regular space curve equipped with the TMOF. There does not exist a generic spatial curve ($\tau \neq 0$) that admits a $\mathbf{T}-\tilde{\mathbf{O}}$ associated partner.

Proof. Assume that there exists a partner curve $\tilde{\gamma}$ which is a $\mathbf{T}-\tilde{\mathbf{O}}$ associated curve of γ . Their spatial relationship is governed by:

$$\tilde{\gamma}(s) = \gamma(s) + \delta(s)\mathbf{T}(s). \quad (8)$$

Differentiating Eq. (8) and utilizing the TMOF derivative formulas yields:

$$\tilde{\gamma}' = (1 + \delta')\mathbf{T} + \delta \frac{\kappa}{\tau} \mathbf{N}. \quad (9)$$

The tangent vector of the associated curve is explicitly given by:

$$\tilde{\mathbf{T}} = \frac{(1 + \delta')\mathbf{T} + \delta \frac{\kappa}{\tau} \mathbf{N}}{\|\tilde{\gamma}'\|}. \quad (10)$$

Taking the second derivative and introducing the auxiliary function $\Phi = (1 + \delta')(\kappa(1 + \delta') + (\delta\kappa)') - \delta\kappa(\delta'' - \delta\kappa^2)$, the cross product is found as:

$$\tilde{\gamma}' \times \tilde{\gamma}'' = \delta^2 \kappa^2 \tau \mathbf{T} - \delta\kappa(1 + \delta')\mathbf{N} + \frac{\Phi}{\tau} \mathbf{B}. \quad (11)$$

Using the definitions in Eq. (2), the binormal and principal normal vectors are obtained as a single system:

$$\begin{aligned} \tilde{\mathbf{B}} &= \frac{\tilde{\tau}}{\|\tilde{\gamma}' \times \tilde{\gamma}''\|} \left(\delta^2 \kappa^2 \tau \mathbf{T} - \delta\kappa(1 + \delta')\mathbf{N} + \frac{\Phi}{\tau} \mathbf{B} \right), \\ \tilde{\mathbf{N}} &= \frac{\tilde{\tau}}{\|\tilde{\gamma}'\| \|\tilde{\gamma}' \times \tilde{\gamma}''\|} \left(-\frac{\delta\kappa\Phi}{\tau^2} \mathbf{T} + \frac{(1 + \delta')\Phi}{\tau} \mathbf{N} + (\delta^3 \kappa^3 + \delta\kappa(1 + \delta')^2) \mathbf{B} \right) \end{aligned} \quad (12)$$

Using the definitions of the moving frame vectors, since $\tilde{\gamma}$ is defined as a $\mathbf{T}-\tilde{\mathbf{O}}$ associated curve, setting the necessary inner product relations $\langle \mathbf{T}, \tilde{\mathbf{T}} \rangle = \langle \tilde{\mathbf{O}}, \tilde{\mathbf{T}} \rangle$, $\langle \mathbf{T}, \tilde{\mathbf{N}} \rangle = \langle \tilde{\mathbf{O}}, \tilde{\mathbf{N}} \rangle$, and $\langle \mathbf{T}, \tilde{\mathbf{B}} \rangle = \langle \tilde{\mathbf{O}}, \tilde{\mathbf{B}} \rangle = 0$ yields

$$\frac{1+\delta'}{\sqrt{(1-\delta\kappa^2)^2+((\delta\kappa)')^2}} = \frac{u}{\sqrt{u^2+(v\tilde{\tau})^2}}, \quad (13)$$

$$\frac{-\delta\kappa\Phi}{\|\tilde{\gamma}' \times \tilde{\gamma}''\| \|\tilde{\gamma}'\|} = \frac{v\tilde{\tau}}{\sqrt{u^2+(v\tilde{\tau})^2}}, \quad (14)$$

$$\frac{\tau\tilde{\tau}\delta^2\kappa^2}{\|\tilde{\gamma}' \times \tilde{\gamma}''\|} = 0. \quad (15)$$

From Eq. (15), it immediately follows that $\tau\tilde{\tau}\delta^2\kappa^2 = 0$. However, in this case, since $\tilde{\mathbf{T}} = \mathbf{T}$ and $\tilde{\mathbf{N}} = \tilde{\mathbf{B}} = \mathbf{0}$, an orthogonal frame cannot be established. Under these conditions, since γ reduces to a straight line, γ has no $\mathbf{T}$ - $\tilde{\mathbf{O}}$ associated curve. $\square$

Definition 2. Let γ and $\tilde{\gamma}$ be regular space curves defined with the TMOF. If the tangent vector $\mathbf{T}(s)$ of the curve γ and the normal-plane vector $\tilde{\mathbf{P}}(s)$ of the curve $\tilde{\gamma}$ are collinear, i.e., $\mathbf{T}(s) = \tilde{\mathbf{P}}(s)$, then $\tilde{\gamma}$ is called a $\mathbf{T}$ - $\tilde{\mathbf{P}}$ associated curve of γ .

Theorem 2. If $\tilde{\gamma}$ is the $\mathbf{T}$ - $\tilde{\mathbf{P}}$ associated curve of γ in the TMOF setting, then the relationship between their modified orthogonal frames is given by:

$$\begin{aligned} \tilde{\mathbf{T}} &= \frac{\mathbf{N}}{x}, \\ \tilde{\mathbf{N}} &= \frac{\tilde{\tau}w}{\sqrt{w^2+x^2}}\mathbf{T} + \frac{\tilde{\tau}}{\sqrt{w^2+x^2}}\mathbf{B}, \\ \tilde{\mathbf{B}} &= \frac{\tilde{\tau}x}{\sqrt{w^2+x^2}}\mathbf{T} - \frac{\tilde{\tau}w}{x\sqrt{w^2+x^2}}\mathbf{B}. \end{aligned}$$

Proof. Since $\tilde{\gamma}$ is defined as the $\mathbf{T}$ - $\tilde{\mathbf{P}}$ associated curve of γ , it directly follows that the tangent vector $\mathbf{T}$ lies strictly within the normal plane of $\tilde{\gamma}$. This requires the orthogonality condition $\langle \mathbf{T}, \tilde{\mathbf{T}} \rangle = 0$. Substituting the explicit form of $\tilde{\mathbf{T}}$ yields:

$$\frac{1+\delta'}{\sqrt{(1+\delta')^2+(\delta\kappa)^2}} = 0 \Rightarrow 1+\delta' = 0.$$

Integrating this differential equation gives the distance function:

$$\delta(s) = -s + c, \quad (16)$$

where c is a constant of integration. Next, evaluating the inner products $\langle \mathbf{T}, \tilde{\mathbf{N}} \rangle = \langle \tilde{\mathbf{P}}, \tilde{\mathbf{N}} \rangle$ and $\langle \mathbf{T}, \tilde{\mathbf{B}} \rangle = \langle \tilde{\mathbf{P}}, \tilde{\mathbf{B}} \rangle$, and substituting $1+\delta' = 0$ dramatically simplifies the equations into the following system:

$$\begin{aligned}\frac{\tilde{\tau}w}{\sqrt{w^2+x^2}} &= \frac{\tilde{\tau}\delta^3\kappa^4}{\|\tilde{\gamma}'\|\|\tilde{\gamma}'\times\tilde{\gamma}''\|}, \\ \frac{\tilde{\tau}x}{\sqrt{w^2+x^2}} &= \frac{\tilde{\tau}\delta^2\kappa^2\tau}{\|\tilde{\gamma}'\times\tilde{\gamma}''\|}.\end{aligned}\quad (17)$$

Under the condition $1 + \delta' = 0$, the fundamental norms reduce to $\|\tilde{\gamma}'\| = \delta\kappa$ and $\|\tilde{\gamma}' \times \tilde{\gamma}''\| = \delta^2\kappa^2\sqrt{\tau^2 + \kappa^2}$. Substituting these norms into Eq. (17) reduces them to the following compact proportional forms:

$$\frac{w}{\sqrt{w^2+x^2}} = \frac{-\kappa}{\sqrt{\tau^2+\kappa^2}} \quad \text{and} \quad \frac{x}{\sqrt{w^2+x^2}} = \frac{\tau}{\sqrt{\tau^2+\kappa^2}}. \quad (18)$$

These strict equalities establish the proportionalities $w = -\kappa$ and $x = \tau$. Since $\kappa > 0$ by our initial geometric assumptions, it necessarily follows that $w < 0$. Substituting these parameters back into the generic frame formulas seamlessly recovers the stated TMOF expressions and completes the proof. $\square$

Corollary 1. As an immediate geometric consequence of Eq. (16), if $\tilde{\gamma}$ is the $\mathbf{T}\text{--}\tilde{\mathbf{P}}$ associated curve of γ in the TMOF setting, then $\tilde{\gamma}$ intrinsically manifests as an involute curve of the base curve γ .

Theorem 3. If $\tilde{\gamma}$ denotes the $\mathbf{T}\text{--}\tilde{\mathbf{P}}$ associated curve of γ in the TMOF setting, then the curvature $\tilde{\kappa}$ and torsion $\tilde{\tau}$ of $\tilde{\gamma}$ are explicitly given by the following formulas:

$$\begin{aligned}\tilde{\kappa} &= \frac{\tau\sqrt{w^2+x^2}}{x\delta\kappa}, \\ \tilde{\tau} &= \frac{\kappa\tau' - \kappa'\tau}{\delta\kappa(\tau^2 + \kappa^2)}.\end{aligned}\quad (19)$$

Proof. Substituting Eq. (16) and the proportionalities established in Eq. (18) directly into the general parameter formulas in Eq. (3) yields the desired curvature and torsion expressions. $\square$

Theorem 4. Let γ be a given base curve and $\tilde{\gamma}$ be its $\mathbf{T}\text{--}\tilde{\mathbf{P}}$ associated curve. The spatial Euclidean distance between the corresponding points of γ and $\tilde{\gamma}$ is given by:

$$d(\gamma, \tilde{\gamma}) = |-s + c|. \quad (20)$$

Proof. The proof follows straightforwardly from the explicitly integrated distance function $\delta(s)$ derived in Eq. (16). $\square$

Definition 3. Let γ and $\tilde{\gamma}$ be regular space curves defined with the TMOF. If the tangent vector $\mathbf{T}(s)$ of the curve γ and the rectifying-plane vector $\tilde{\mathbf{R}}(s)$ of the curve $\tilde{\gamma}$ are collinear, i.e., $\mathbf{T}(s) = \tilde{\mathbf{R}}(s)$, then $\tilde{\gamma}$ is called a $\mathbf{T}\text{--}\tilde{\mathbf{R}}$ associated curve of γ .

Theorem 5. If $\tilde{\gamma}$ is the $\mathbf{T}$ – $\tilde{\mathbf{R}}$ associated curve of γ in the TMOF setting, then the relationship between their modified orthogonal frames is given by:

$$\begin{aligned}\tilde{\mathbf{T}} &= \frac{y}{\sqrt{y^2 + (z\tilde{\tau})^2}} \mathbf{T} + \frac{z\tilde{\tau}}{\tau\sqrt{y^2 + (z\tilde{\tau})^2}} \mathbf{N}, \\ \tilde{\mathbf{N}} &= \frac{\tilde{\tau}}{\tau} \mathbf{B}, \\ \tilde{\mathbf{B}} &= \frac{z\tilde{\tau}^2}{\sqrt{y^2 + (z\tilde{\tau})^2}} \mathbf{T} - \frac{y\tilde{\tau}}{\tau\sqrt{y^2 + (z\tilde{\tau})^2}} \mathbf{N}.\end{aligned}$$

Proof. By the definition of the $\mathbf{T}$ – $\tilde{\mathbf{R}}$ associated curve, setting the inner product relations $\langle \mathbf{T}, \tilde{\mathbf{T}} \rangle = \langle \tilde{\mathbf{R}}, \tilde{\mathbf{T}} \rangle$, $\langle \mathbf{T}, \tilde{\mathbf{N}} \rangle = \langle \tilde{\mathbf{R}}, \tilde{\mathbf{N}} \rangle = 0$, and $\langle \mathbf{T}, \tilde{\mathbf{B}} \rangle = \langle \tilde{\mathbf{R}}, \tilde{\mathbf{B}} \rangle$ alongside the frame vectors yields the following system:

$$\frac{1+\delta'}{\sqrt{(1+\delta')^2 + (\delta\kappa)^2}} = \frac{y}{\sqrt{y^2 + (z\tilde{\tau})^2}} \Rightarrow 1 + \delta' = \frac{y\delta\kappa}{z\tilde{\tau}}, \quad (21)$$

$$-\frac{\tilde{\tau}\delta\kappa\Phi}{\|\tilde{\gamma}' \times \tilde{\gamma}''\| \|\tilde{\gamma}'\|} = 0, \quad (22)$$

$$\frac{\tau\tilde{\tau}\delta^2\kappa^2}{\|\tilde{\gamma}' \times \tilde{\gamma}''\|} = \frac{z\tilde{\tau}}{\sqrt{y^2 + (z\tilde{\tau})^2}}, \quad (23)$$

where $\Phi = (1 + \delta')(\kappa(1 + \delta') + (\delta\kappa)') - \delta\kappa(\delta'' - \delta\kappa^2)$. Since the geometric parameters $\tilde{\tau}$, δ , and κ are non-zero, Eq. (22) necessitates that the auxiliary function must identically vanish ($\Phi = 0$). Under this condition, the norm of the cross product simplifies to:

$$\|\tilde{\gamma}' \times \tilde{\gamma}''\| = \tau\delta\kappa\sqrt{(\delta\kappa)^2 + (1 + \delta')^2}. \quad (24)$$

Consequently, by simultaneously evaluating the relations in Eqs. (21)–(24) in conjunction with Eq. (10) and Eq. (12), the proof is successfully completed. $\square$

Theorem 6. If $\tilde{\gamma}$ is the $\mathbf{T}$ – $\tilde{\mathbf{R}}$ associated curve of γ , the curvature $\tilde{\kappa}$ and torsion $\tilde{\tau}$ are explicitly given by:

$$\tilde{\kappa} = \frac{\tau\delta\kappa}{(1+\delta')^2 + (\delta\kappa)^2}, \quad \tilde{\tau} = \frac{\Omega_1}{(\tau\delta\kappa)^2((1+\delta')^2 + (\delta\kappa)^2)}, \quad (25)$$

where the auxiliary function Ω_1 is defined as:

$$\begin{aligned}\Omega_1 &= \tau\delta^2\kappa^2 \left(\delta'''' - 3\delta'\kappa^2 - 3\delta\kappa\kappa' - \kappa^2 + \delta\kappa^2 \frac{\tau'}{\tau} \right) \\ &\quad - \tau^2\delta\kappa(1 + \delta')(3\delta''\frac{\kappa}{\tau} - \delta\frac{\kappa^3}{\tau} - \delta\kappa\tau \\ &\quad + (1 + 3\delta')\left(\frac{\kappa}{\tau}\right)' + \delta\left(\frac{\kappa}{\tau}\right)'' + (1 + 2\delta')\frac{\kappa\tau'}{\tau^2} + \delta\frac{\tau'}{\tau}\left(\frac{\kappa}{\tau}\right)'. \end{aligned} \quad (26)$$

Proof. Substituting the derived relations into the standard curvature formula yields $\tilde{\kappa}$. To find the torsion, differentiating Eq. (8) three times gives the third derivative:

$$\begin{aligned}\tilde{\gamma}''' &= \left(\delta''' - 3\delta'\kappa^2 - 3\delta\kappa\kappa' - \kappa^2 + \delta\kappa^2\frac{\tau'}{\tau} \right) \mathbf{T} \\ &\quad + (\kappa + 3\delta'\kappa + 2\delta\kappa') \mathbf{B} \\ &\quad + (3\delta''\frac{\kappa}{\tau} - \delta\frac{\kappa^3}{\tau} - \delta\kappa\tau + (1 + 3\delta')\left(\frac{\kappa}{\tau}\right)' + \delta\left(\frac{\kappa}{\tau}\right)'' \\ &\quad + (1 + 2\delta')\frac{\kappa\tau'}{\tau^2} + \delta\frac{\tau'}{\tau}\left(\frac{\kappa}{\tau}\right)') \mathbf{N}.\end{aligned}\tag{27}$$

Using Eq. (27) in the torsion formula completes the proof. $\square$

Theorem 7. Let γ and $\tilde{\gamma}$ be the associated partner curves. The distance function $\delta(s)$ between the corresponding points in E^3 is given by:

$$d(\gamma, \tilde{\gamma}) = \left| e^{\int \frac{y\kappa}{z\tilde{\tau}} ds} \left[-\int e^{-\int \frac{y\kappa}{z\tilde{\tau}} ds} ds + c_1 \right] \right|,\tag{28}$$

where c_1 is an integration constant.

Proof. From the geometric relation in Eq. (21), we obtain the following first-order linear differential equation:

$$\delta' - \left(\frac{y\kappa}{z\tilde{\tau}} \right) \delta = -1.\tag{29}$$

The general solution of this ODE is found by employing the integrating factor $\mu(s) = \exp\left(-\int \frac{y\kappa}{z\tilde{\tau}} ds\right)$. Integrating the transformed equation $(\mu\delta)' = -\mu$ with respect to s directly leads to:

$$\delta(s) = \frac{-\int \mu(s) ds + c_1}{\mu(s)}.$$

Substituting the explicit form of $\mu(s)$ back into the solution successfully completes the proof. $\square$

Principal Normal-Associated Curves in the TMOF Setting

We analyze associated partner curves generated by normal displacement ($V = \mathbf{N}$) within the TMOF setting. Maintaining the direction fields $\tilde{\mathbf{O}}$, $\tilde{\mathbf{P}}$, and $\tilde{\mathbf{R}}$ as previously established, we determine the geometric conditions under which the principal normal $\mathbf{N}(s)$ aligns with each distinguished planar direction of the partner curve.

Definition 4. Let γ and $\tilde{\gamma}$ be regular space curves defined with the TMOF. If the principal normal vector $\mathbf{N}(s)$ of the curve γ and the osculating-plane vector $\tilde{\mathbf{O}}(s)$ of the curve $\tilde{\gamma}$ are collinear, i.e., $\mathbf{N}(s) = \tilde{\mathbf{O}}(s)$, then $\tilde{\gamma}$ is called an $\mathbf{N}-\tilde{\mathbf{O}}$ associated curve of γ .

Theorem 8. If $\tilde{\gamma}$ is the $\mathbf{N}-\tilde{\mathbf{O}}$ associated curve of γ in the TMOF setting, then the relationship between their modified orthogonal frames is given by:

$$\begin{aligned}\tilde{\mathbf{T}} &= \frac{u}{\sqrt{u^2 + (v\tilde{\tau})^2}} \left(\frac{1 - \delta\kappa\tau}{\tau(\delta\tau)'} \mathbf{T} + \frac{1}{\tau^2} \mathbf{N} + \frac{\delta\tau}{\tau(\delta\tau)'} \mathbf{B} \right), \\ \tilde{\mathbf{N}} &= \frac{v\tilde{\tau}^2}{\tau\sqrt{u^2 + (v\tilde{\tau})^2}} \left(\frac{-(\delta\tau)' \mathbf{F}}{(1 - \delta\kappa\tau)\mathbf{F} - \delta\tau^2 \mathbf{D}} \mathbf{T} + \frac{1}{\tau} \mathbf{N} \right. \\ &\quad \left. + \frac{(\delta\tau)' \mathbf{D}}{\tau((1 - \delta\kappa\tau)\mathbf{F} - \delta\tau^2 \mathbf{D})} \mathbf{B} \right), \\ \tilde{\mathbf{B}} &= \frac{v\tilde{\tau}^2(\delta\tau)'}{u\tau((1 - \delta\kappa\tau)\mathbf{F} - \delta\tau^2 \mathbf{D})} (\tau \mathbf{D} \mathbf{T} + \mathbf{F} \mathbf{B}),\end{aligned}$$

where $\mathbf{D}$ and $\mathbf{F}$ are the coefficients,

$$\begin{aligned}\mathbf{D} &= \tau(\delta\tau)'(2\delta'\tau + 3\delta\tau') \\ &\quad - \delta\tau^2 \left((1 - \delta\kappa\tau) \frac{\kappa}{\tau} + \delta'' + \delta \frac{\tau''}{\tau} + 2\delta' \frac{\tau'}{\tau} - \delta\tau^2 \right), \\ \mathbf{F} &= (1 - \delta\kappa\tau) \left((1 - \delta\kappa\tau) \frac{\kappa}{\tau} + \delta'' + \delta \frac{\tau''}{\tau} + 2\delta' \frac{\tau'}{\tau} - \delta\tau^2 \right) \\ &\quad + \frac{(\delta\tau)'}{\tau} (2\delta'\kappa\tau + 2\delta\kappa\tau' + \delta\kappa'\tau).\end{aligned}$$

Proof. Given that γ and $\tilde{\gamma}$ form an $\mathbf{N}-\tilde{\mathbf{O}}$ associated pair, their spatial relationship is defined as:

$$\tilde{\gamma}(s) = \gamma(s) + \delta(s)\mathbf{N}(s). \quad (30)$$

Differentiating Eq. (30) and utilizing the TMOF derivative formulas provided in Eq. (1) yields:

$$\tilde{\mathbf{T}} = \frac{(1 - \delta\kappa\tau)\mathbf{T} + \frac{(\delta\tau)'}{\tau}\mathbf{N} + \delta\tau\mathbf{B}}{\sqrt{(1 - \delta\kappa\tau)^2 + ((\delta\tau)')^2 + \delta^2\tau^4}}. \quad (31)$$

Taking the second derivative of Eq. (30) and applying Eq. (1) again, we obtain:

$$\begin{aligned}
\tilde{\gamma}'' &= -(2\delta'\kappa\tau + 2\delta\kappa\tau' + \delta\kappa'\tau)\mathbf{T} \\
&+ \left((1 - \delta\kappa\tau)\frac{\kappa}{\tau} + \delta'' + \delta\frac{\tau''}{\tau} + 2\delta'\frac{\tau'}{\tau} - \delta\tau^2 \right)\mathbf{N} \\
&+ (2\delta'\tau + 3\delta\tau')\mathbf{B}.
\end{aligned} \tag{32}$$

The cross product of $\tilde{\gamma}'$ and Eq. (32) is expressed in the following compact form:

$$\tilde{\gamma}' \times \tilde{\gamma}'' = \tau\mathbf{D}\mathbf{T} + \mathbf{E}\mathbf{N} + \mathbf{F}\mathbf{B}, \tag{33}$$

where the auxiliary scalar functions $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{F}$ are explicitly defined as:

$$\begin{aligned}
\mathbf{D} &= (\delta\tau)'(2\delta'\tau + 3\delta\tau') \\
&\quad - \delta\tau^2 \left((1 - \delta\kappa\tau)\frac{\kappa}{\tau} + \delta'' + \delta\frac{\tau''}{\tau} + 2\delta'\frac{\tau'}{\tau} - \delta\tau^2 \right), \\
\mathbf{E} &= -(1 - \delta\kappa\tau)(2\delta'\tau + 3\delta\tau') - \delta\tau(2\delta'\kappa\tau + 2\delta\kappa\tau' + \delta\kappa'\tau), \\
\mathbf{F} &= (1 - \delta\kappa\tau) \left((1 - \delta\kappa\tau)\frac{\kappa}{\tau} + \delta'' + \delta\frac{\tau''}{\tau} + 2\delta'\frac{\tau'}{\tau} - \delta\tau^2 \right) \\
&\quad + \frac{(\delta\tau)'}{\tau} (2\delta'\kappa\tau + 2\delta\kappa\tau' + \delta\kappa'\tau).
\end{aligned}$$

Consequently, the norm of the cross product is given by:

$$\|\tilde{\gamma}' \times \tilde{\gamma}''\| = \tau\sqrt{\mathbf{D}^2 + \mathbf{E}^2 + \mathbf{F}^2}. \tag{34}$$

With the aid of Eq. (34) and Eq. (2), we derive the expressions for the normal and binormal fields $\tilde{\mathbf{N}}$ and $\tilde{\mathbf{B}}$ as follows:

$$\begin{aligned}
\tilde{\mathbf{B}} &= \tilde{\tau} \frac{\tau\mathbf{D}\mathbf{T} + \mathbf{E}\mathbf{N} + \mathbf{F}\mathbf{B}}{\tau\sqrt{\mathbf{D}^2 + \mathbf{E}^2 + \mathbf{F}^2}}, \\
\tilde{\mathbf{N}} &= \left(\frac{\tau^2\delta\tau\tilde{\tau}\mathbf{E} - \tau(\delta\tau)'\tilde{\tau}\mathbf{F}}{\tau\|\tilde{\gamma}'\|\sqrt{\mathbf{D}^2 + \mathbf{E}^2 + \mathbf{F}^2}} \right)\mathbf{T} + \left(\frac{(1 - \delta\kappa\tau)\tilde{\tau}\mathbf{F} - \tau^2\delta\tilde{\tau}\mathbf{D}}{\tau\|\tilde{\gamma}'\|\sqrt{\mathbf{D}^2 + \mathbf{E}^2 + \mathbf{F}^2}} \right)\mathbf{N} \\
&\quad + \left(\frac{(\delta\tau)'\tilde{\tau}\mathbf{D} - (1 - \delta\kappa\tau)\tilde{\tau}\mathbf{E}}{\tau\|\tilde{\gamma}'\|\sqrt{\mathbf{D}^2 + \mathbf{E}^2 + \mathbf{F}^2}} \right)\mathbf{B}.
\end{aligned} \tag{35}$$

By the definition of the $\mathbf{N}$ - $\tilde{\mathbf{O}}$ associated curve, we have the relation $\langle \mathbf{N}, \tilde{\mathbf{T}} \rangle = \langle \tilde{\mathbf{O}}, \tilde{\mathbf{T}} \rangle$. Applying this using Eq. (31) and Eq. (5) yields:

$$\frac{\tau(\delta\tau)'}{\sqrt{(1 - \delta\kappa\tau)^2 + ((\delta\tau)')^2 + \delta^2\tau^4}} = \frac{u}{\sqrt{u^2 + (v\tilde{\tau})^2}}. \tag{36}$$

Similarly, evaluating the inner product $\langle \mathbf{N}, \tilde{\mathbf{N}} \rangle = \langle \tilde{\mathbf{O}}, \tilde{\mathbf{N}} \rangle$ provides:

$$\frac{v\tilde{\tau}}{\sqrt{u^2 + (v\tilde{\tau})^2}} = \frac{\tau((1 - \delta\kappa\tau)\mathbf{F} - \tau^2\delta\mathbf{D})}{\sqrt{(1 - \delta\kappa\tau)^2 + ((\delta\tau)')^2 + \delta^2\tau^4}\sqrt{\mathbf{D}^2 + \mathbf{E}^2 + \mathbf{F}^2}}, \tag{37}$$

and finally, the orthogonality condition $\langle \mathbf{N}, \tilde{\mathbf{B}} \rangle = \langle \tilde{\mathbf{O}}, \tilde{\mathbf{B}} \rangle = 0$ gives:

$$\frac{\tau \tilde{\tau} \mathbf{E}}{\sqrt{\mathbf{D}^2 + \mathbf{E}^2 + \mathbf{F}^2}} = 0. \quad (38)$$

Since $\tau, \tilde{\tau} \neq 0$, Eq. (38) implies $\mathbf{E} = 0$. Substituting Eqs. (36)–(38) into Eqs. (31) and (35) completes the proof. $\square$

Corollary 2. Let $\tilde{\gamma}$ be the $\mathbf{N}$ – $\tilde{\mathbf{O}}$ associated curve of γ in the TMOF setting. If the base curve γ is chosen to have a non-zero constant torsion ($\tau = \text{constant}$), then the distance function $\delta(s)$ between the associated curves is explicitly determined by the curvature $\kappa(s)$ of the base curve as $\delta(s) = \frac{2}{\tau \kappa(s) + c}$, where c is a constant of integration.

Proof. From the orthogonality condition in Eq. (38), we established that $\mathbf{E} = 0$. Under the assumption of constant torsion ($\tau' = 0$), the explicit equation for $\mathbf{E}$ reduces to:

$$2\delta' + \delta^2 \kappa' \tau = 0. \quad (39)$$

Eq. (39) is a separable first-order differential equation, which can be rearranged as $\frac{2\delta'}{\delta^2} = -\tau \kappa'$. Integrating both sides with respect to the arc-length parameter s directly yields the desired result. $\square$

Theorem 9. If $\tilde{\gamma}$ is the $\mathbf{N}$ – $\tilde{\mathbf{O}}$ associated curve of γ in the TMOF setting, then the curvature $\tilde{\kappa}$ and torsion $\tilde{\tau}$ of the partner curve $\tilde{\gamma}$ are explicitly given by:

$$\begin{aligned} \tilde{\kappa} &= \frac{u^4 \left((1 - \delta \kappa \tau) \mathbf{F} - \delta \tau^2 \mathbf{D} \right)}{v \tau^2 \tilde{\tau} (\delta \tau)' (u^2 + (v \tilde{\tau})^2)^{\frac{3}{2}}}, \\ \tilde{\tau} &= 3 \sqrt{\frac{\Omega_2}{\left(\frac{u \tau \left((1 - \delta \kappa \tau) \mathbf{F} - \delta \tau^2 \mathbf{D} \right)}{v (\delta \tau)'} \right)^2}}, \end{aligned} \quad (40)$$

where,

$$\begin{aligned} \Omega_2 &= \tau \mathbf{D} \left(-3\delta'' \kappa \tau - 3\delta' \kappa' \tau - 6\delta' \kappa \tau' - \delta \kappa'' \tau - 3\delta \kappa' \tau' - 3\delta \kappa \tau'' - \kappa^2 + \delta \kappa^3 \tau + \delta \kappa \tau^3 \right) \\ &+ \tau^3 \mathbf{F} \left(\frac{\kappa}{\tau} - \delta \kappa^2 + 3\delta'' + 9\delta' \frac{\tau'}{\tau} + 4\delta \frac{\tau''}{\tau} - \delta \tau^2 + 3\delta \left(\frac{\tau'}{\tau} \right)^2 \right), \end{aligned}$$

Proof. Substituting Eqs. (36)–(38) into the standard curvature formula yields $\tilde{\kappa}$. Taking the third derivative of Eq. (30) provides:

$$\begin{aligned}
\tilde{\gamma}''' = & \left(-3\delta''\kappa\tau - 3\delta'\kappa'\tau - 6\delta'\kappa\tau' - \delta\kappa''\tau - 3\delta\kappa'\tau' - 3\delta\kappa\tau'' - \kappa^2 + \delta\kappa^3\tau + \delta\kappa\tau^3 \right) \mathbf{T} \\
& + \left(\frac{\kappa'}{\tau} - 3\delta'\kappa^2 - 3\delta\kappa\kappa' - 3\delta\kappa^2 \frac{\tau'}{\tau} + \delta''' + 3\delta'' \frac{\tau'}{\tau} + 3\delta' \frac{\tau''}{\tau} + \delta \frac{\tau'''}{\tau} - 3\delta'\tau^2 - 6\delta\tau\tau' \right) \mathbf{N} \\
& + \tau \left(\frac{\kappa}{\tau} - \delta\kappa^2 + 3\delta'' + 9\delta' \frac{\tau'}{\tau} + 4\delta \frac{\tau''}{\tau} - \delta\tau^2 + 3\delta \left(\frac{\tau'}{\tau} \right)^2 \right) \mathbf{B}.
\end{aligned} \tag{41}$$

Using Eq. (41) within the standard torsion formula provides $\tilde{\tau}$. $\square$

Definition 5. Let γ and $\tilde{\gamma}$ be regular space curves defined with the TMOF. If the principal normal vector $\mathbf{N}(s)$ of the curve γ and the normal-plane vector $\tilde{\mathbf{P}}(s)$ of the curve $\tilde{\gamma}$ are collinear, i.e., $\mathbf{N}(s) = \tilde{\mathbf{P}}(s)$, then $\tilde{\gamma}$ is called an $\mathbf{N}$ - $\tilde{\mathbf{P}}$ associated curve of γ .

Theorem 10. If $\tilde{\gamma}$ is the $\mathbf{N}$ - $\tilde{\mathbf{P}}$ associated curve of γ in the TMOF setting, then the relationship between their modified orthogonal frames is given by:

$$\begin{aligned}
\tilde{\mathbf{T}} &= \frac{(1 - \delta\kappa\tau)}{\sqrt{(1 - \delta\kappa\tau)^2 + \tau^2(\delta\tau)^2}} \mathbf{T} + \frac{\delta\tau}{\sqrt{(1 - \delta\kappa\tau)^2 + \tau^2(\delta\tau)^2}} \mathbf{B}, \\
\tilde{\mathbf{N}} &= \frac{\tilde{\tau}}{\sqrt{w^2 + x^2}} \left(\frac{\tau\delta x}{\sqrt{(1 - \delta\kappa\tau)^2 + \tau^2(\delta\tau)^2}} \mathbf{T} + \frac{w}{\tau^2} \mathbf{N} \right. \\
&\quad \left. - \frac{x(1 - \delta\kappa\tau)}{\tau^2 \sqrt{(1 - \delta\kappa\tau)^2 + \tau^2(\delta\tau)^2}} \mathbf{B} \right), \\
\tilde{\mathbf{B}} &= \frac{\tilde{\tau}(\tau\mathbf{d}\mathbf{T} + \mathbf{e}\mathbf{N} + \mathbf{f}\mathbf{B})}{\tau\sqrt{\mathbf{d}^2 + \mathbf{e}^2 + \mathbf{f}^2}},
\end{aligned}$$

where $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{f}$ are the reduced coefficients,

$$\begin{aligned}
\mathbf{d} &= -\delta\tau^3 \left(\frac{\kappa}{\tau} - \delta\kappa^2 - \delta\tau^2 \right), \quad \mathbf{e} = -\delta\tau'(1 - \delta\kappa\tau) - \delta^2\tau^2\kappa', \\
\mathbf{f} &= (1 - \delta\kappa\tau) \left(\frac{\kappa}{\tau} - \delta\kappa^2 - \delta\tau^2 \right).
\end{aligned}$$

Proof. By the definition of the $\mathbf{N}$ - $\tilde{\mathbf{P}}$ associated curve, the collinearity and orthogonality conditions impose the following inner product relations:

$$\langle \mathbf{N}, \tilde{\mathbf{N}} \rangle = \langle \tilde{\mathbf{P}}, \tilde{\mathbf{N}} \rangle, \quad \langle \mathbf{N}, \tilde{\mathbf{B}} \rangle = \langle \tilde{\mathbf{P}}, \tilde{\mathbf{B}} \rangle, \quad \langle \mathbf{N}, \tilde{\mathbf{T}} \rangle = \langle \tilde{\mathbf{P}}, \tilde{\mathbf{T}} \rangle = 0.$$

Evaluating these specific inner products with respect to the general TMOF components yields the following system of equations:

$$\begin{aligned}
\frac{w}{\sqrt{w^2 + x^2}} &= \frac{\tau \left((1 - \delta\kappa\tau)F - \tau^2\delta D \right)}{\sqrt{(1 - \delta\kappa\tau)^2 + (\delta\tau')^2 + \delta^2\tau^4} \sqrt{D^2 + E^2 + F^2}}, \\
\frac{x}{\sqrt{w^2 + x^2}} &= \frac{\tau E}{\sqrt{D^2 + E^2 + F^2}}, \\
\frac{\tau(\delta\tau)'}{\sqrt{(1 - \delta\kappa\tau)^2 + (\delta\tau')^2 + \delta^2\tau^4}} &= 0,
\end{aligned} \tag{42}$$

Since $\tau \neq 0$, Eq. (42) implies $(\delta\tau)' = 0$. Substituting this constraint into $\mathbf{D}, \mathbf{E}, \mathbf{F}$ yields the reduced coefficients $\mathbf{d}, \mathbf{e}, \mathbf{f}$, which define $\tilde{\mathbf{T}}$ and $\tilde{\mathbf{N}}$ to complete the proof. $\square$

Corollary 3. In the Torsion-Modified Orthogonal Frame (TMOF) setting, if $\tilde{\gamma}$ is an $\mathbf{N}-\tilde{\mathbf{P}}$ associated curve of γ , then the distance function between the curves is given by $\delta(s) = \frac{c}{\tau(s)}$, ($c \in \mathbb{R}$). Since the partner curve $\tilde{\gamma}$ follows the reference curve at a constant normal distance throughout the motion, the $\mathbf{N}-\tilde{\mathbf{P}}$ pairing in TMOF constitutes a Bertrand and Mannheim curve pair (Lone et al., 2019; Elsharkawy and Elshenhab, 2022; Demirkıran, 2022).

Theorem 11. If $\tilde{\gamma}$ denotes the $\mathbf{N}-\tilde{\mathbf{P}}$ associated curve of γ in the TMOF setting, then the curvature $\tilde{\kappa}$ and torsion $\tilde{\tau}$ of $\tilde{\gamma}$ are given by:

$$\tilde{\kappa} = \frac{\tau^2 w^3 \mathbf{e}^4 \sqrt{w^2 + x^2}}{x^4 ((1 - \delta\kappa\tau)\mathbf{f} - \delta\tau^2 \mathbf{d})^3}, \quad \tilde{\tau} = \frac{\Omega_3 x^2}{\mathbf{e}^2 \tau^4 (w^2 + x^2)}, \tag{43}$$

where the auxiliary function Ω_3 is defined as:

$$\begin{aligned}
\Omega_3 &= \tau \mathbf{d} \left(-3\delta''\kappa\tau + \delta'\kappa'\tau - 5\delta'\kappa\tau' - 2\delta\kappa''\tau - \kappa^2 + \kappa^3\delta\tau + \delta\kappa\tau^2 \right) \\
&+ \mathbf{e}\tau^2 \left(\frac{\kappa'}{\tau} - 2\delta\kappa\kappa' + \delta\tau\tau' \right) + f_2\tau^3 \left(\frac{\kappa}{\tau} - \delta\kappa^2 - \delta\tau^2 + \delta\frac{\tau''}{\tau} \right),
\end{aligned}$$

Proof. Substituting $(\delta\tau)' = 0$ from Eq. (42) and the reduced TMOF components into the standard curvature and torsion formulas gives the result. $\square$

Definition 6. Let γ and $\tilde{\gamma}$ be regular space curves defined with the Torsion-Modified Orthogonal Frame (TMOF). If the principal normal vector $\mathbf{N}(s)$ of the curve γ and the rectifying-plane vector $\tilde{\mathbf{R}}(s)$ of the curve $\tilde{\gamma}$ are collinear, i.e., $\mathbf{N}(s) = \tilde{\mathbf{R}}(s)$, then $\tilde{\gamma}$ is called an $\mathbf{N}-\tilde{\mathbf{R}}$ associated curve of γ .

Theorem 12. If $\tilde{\gamma}$ is the $\mathbf{N}-\tilde{\mathbf{R}}$ associated curve of γ in the TMOF setting, then the relationship between their modified orthogonal frames is given by:

$$\begin{aligned}
\tilde{\mathbf{T}} &= \frac{y}{\sqrt{y^2 + (z\tilde{\tau})^2}} \left(\frac{1 - \delta\kappa\tau}{\tau(\delta\tau)'} \mathbf{T} + \frac{1}{\tau^2} \mathbf{N} + \frac{\delta\tau}{\tau(\delta\tau)'} \mathbf{B} \right), \\
\tilde{\mathbf{N}} &= \frac{yz\tilde{\tau}}{y^2 + (z\tilde{\tau})^2} \left(\left(\frac{\delta\tau}{(\delta\tau)'} - \frac{\mathbf{F}}{\tau^2 \mathbf{E}} \right) \mathbf{T} - \left(\frac{1 - \delta\kappa\tau}{\tau^3(\delta\tau)'} - \frac{\mathbf{D}}{\tau^3 \mathbf{E}} \right) \mathbf{B} \right), \\
\tilde{\mathbf{B}} &= \frac{z\tilde{\tau}^2}{\sqrt{y^2 + (z\tilde{\tau})^2}} \left(\frac{\mathbf{D}}{\mathbf{E}\tau} \mathbf{T} + \frac{1}{\tau^2} \mathbf{N} + \frac{\mathbf{F}}{\tau^2 \mathbf{E}} \mathbf{B} \right).
\end{aligned}$$

Proof. By the definition of the $\mathbf{N}$ - $\tilde{\mathbf{R}}$ associated curve, we obtain the following inner product relations:

$$\langle \mathbf{N}, \tilde{\mathbf{T}} \rangle = \langle \tilde{\mathbf{R}}, \tilde{\mathbf{T}} \rangle, \quad \langle \mathbf{N}, \tilde{\mathbf{B}} \rangle = \langle \tilde{\mathbf{R}}, \tilde{\mathbf{B}} \rangle, \quad \langle \mathbf{N}, \tilde{\mathbf{N}} \rangle = \langle \tilde{\mathbf{R}}, \tilde{\mathbf{N}} \rangle = 0.$$

Evaluating these inner products yields the following system:

$$\begin{aligned}
\frac{y}{\sqrt{y^2 + (z\tilde{\tau})^2}} &= \frac{\tau(\delta\tau)'}{\sqrt{(1 - \delta\kappa\tau)^2 + (\delta\tau)'^2 + \delta^2\tau^4}}, \\
\frac{z\tilde{\tau}}{\sqrt{y^2 + (z\tilde{\tau})^2}} &= \frac{\tau \mathbf{E}}{\sqrt{\mathbf{D}^2 + \mathbf{E}^2 + \mathbf{F}^2}}, \\
\frac{\tau((1 - \delta\kappa\tau)\mathbf{F} - \delta\tau^2 \mathbf{D})}{\sqrt{(1 - \delta\kappa\tau)^2 + (\delta\tau)'^2 + \delta^2\tau^4} \sqrt{\mathbf{D}^2 + \mathbf{E}^2 + \mathbf{F}^2}} &= 0,
\end{aligned} \tag{44}$$

Substituting the condition from Eq. (44) into Eqs. (31) and (35) yields the reduced vectors $\tilde{\mathbf{T}}$, $\tilde{\mathbf{N}}$, and $\tilde{\mathbf{B}}$. $\square$

Theorem 13. If $\tilde{\gamma}$ denotes the $\mathbf{N}$ - $\tilde{\mathbf{R}}$ associated curve of γ in the TMOF setting, then the curvature $\tilde{\kappa}$ and torsion $\tilde{\tau}$ are given by:

$$\begin{aligned}
\tilde{\kappa} &= \frac{\mathbf{E}y^3}{\tau z \tilde{\tau} ((\delta\tau)')^3 (y^2 + (z\tilde{\tau})^2)}, \\
\tilde{\tau} &= \frac{\Omega_4 z^2 \pm \sqrt{\Omega_4^2 z^4 - 4\tau^8 \mathbf{E}^4 y^2 z^2}}{2\tau^4 \mathbf{E}^2 z^2},
\end{aligned} \tag{45}$$

where the auxiliary function Ω_4 is:

$$\begin{aligned}
\Omega_4 &= \tau \mathbf{D} \left(-4\delta''\kappa\tau - 3\delta'\kappa'\tau - 7\delta'\kappa\tau' - 2\delta\kappa''\tau - 4\delta\kappa'\tau' - \delta\kappa\tau'' - \kappa^2 + \kappa^3\delta\tau + \delta\kappa\tau^2 \right) \\
&+ \tau^2 \mathbf{E} \left(\frac{\kappa'}{\tau} - \delta\kappa\kappa' + \delta''' + 3\delta'' \frac{\tau'}{\tau} + 3\delta' \frac{\tau''}{\tau} + \delta \frac{\tau'''}{\tau} - 3\delta'\tau^2 - 3\delta\tau\tau' - \right. \\
&\quad \left. (2\delta'\kappa\tau + 2\delta\kappa\tau' + \delta\kappa'\tau) \frac{\kappa}{\tau} + (2\delta'\tau + 3\delta\tau')\tau \right) \\
&+ \tau^2 \mathbf{F} \left(\frac{\kappa}{\tau} - \delta\kappa^2 + 3\delta'' + 9\delta' \frac{\tau'}{\tau} + 4\delta \frac{\tau''}{\tau} - \delta\tau^2 + 3\delta \left(\frac{\tau'}{\tau} \right)^2 \right),
\end{aligned}$$

Proof. Substituting Eq. (44) and Eq. (41) into the standard curvature and torsion formulas yields $\tilde{\kappa}$ and a quadratic equation for $\tilde{\tau}$. Solving this

quadratic analytically gives $\tilde{\tau}$, provided $\Omega_4^2 z^4 - 4\tau^8 \mathbf{E}^4 y^2 z^2 \geq 0$ for real solutions. $\square$

Corollary 4. Let the base curve γ have non-zero constant curvature and constant torsion ($\kappa, \tau = \text{constant}$). Then, the distance function $\delta(s)$ satisfies the following second-order non-linear differential equation:

$$\delta'' + \frac{2\tau(\kappa(1 - \delta\kappa\tau) - \delta\tau^3)}{(1 - \delta\kappa\tau)^2 + \delta^2\tau^4}(\delta')^2 + \left((1 - \delta\kappa\tau)\frac{\kappa}{\tau} - \delta\tau^2\right) = 0.$$

Proof. Since κ and τ are constant, the coefficients **D** and **E** reduce to

$$\begin{aligned} \mathbf{D} &= 2(\delta')^2\tau^2 - \delta\tau^2 \left((1 - \delta\kappa\tau)\frac{\kappa}{\tau} + \delta'' - \delta\tau^2 \right), \\ \mathbf{E} &= (1 - \delta\kappa\tau) \left((1 - \delta\kappa\tau)\frac{\kappa}{\tau} + \delta'' - \delta\tau^2 \right) + 2(\delta')^2\kappa\tau. \end{aligned}$$

Substituting these simplified expressions into Eq. (44) and setting it equal to zero gives

$$\begin{aligned} (1 - \delta\kappa\tau) \left((1 - \delta\kappa\tau) \left((1 - \delta\kappa\tau)\frac{\kappa}{\tau} + \delta'' - \delta\tau^2 \right) + 2(\delta')^2\kappa\tau \right) \\ - \delta\tau^2 \left(2(\delta')^2\tau^2 - \delta\tau^2 \left((1 - \delta\kappa\tau)\frac{\kappa}{\tau} + \delta'' - \delta\tau^2 \right) \right) = 0. \end{aligned}$$

Expanding the products and grouping the terms with respect to $\left((1 - \delta\kappa\tau)\frac{\kappa}{\tau} + \delta'' - \delta\tau^2 \right)$ and $(\delta')^2$, we arrive directly at the stated second-order differential equation. $\square$

Binormal-Associated Curves in the TMOF Setting

We analyze associated partner curves generated by binormal displacement ($V = \mathbf{B}$) within the TMOF setting. Our objective is to determine the explicit geometric conditions under which the binormal vector $\mathbf{B}(s)$ of the base curve aligns with each distinguished planar direction of its partner trajectory.

Definition 7. Let γ and $\tilde{\gamma}$ be regular space curves defined with the Torsion-Modified Orthogonal Frame (TMOF). If the binormal vector $\mathbf{B}(s)$ of the curve γ and the osculating-plane vector $\tilde{\mathbf{O}}(s)$ of the curve $\tilde{\gamma}$ are collinear, i.e., $\mathbf{B}(s) = \tilde{\mathbf{O}}(s)$, then $\tilde{\gamma}$ is called a $\mathbf{B}$ - $\tilde{\mathbf{O}}$ associated curve of γ .

Theorem 14. If $\tilde{\gamma}$ is the $\mathbf{B}$ - $\tilde{\mathbf{O}}$ associated curve of γ in the TMOF setting, then the relationship between their modified orthogonal frames is given by:

$$\begin{aligned}\tilde{\mathbf{T}} &= \frac{u}{\sqrt{u^2 + (v\tilde{\tau})^2}} \left(\frac{\mathbf{T}}{\tau(\delta\tau)'} - \frac{\delta}{(\delta\tau)'} \mathbf{N} + \frac{\mathbf{B}}{\tau^2} \right), \\ \tilde{\mathbf{N}} &= \frac{v\tilde{\tau}^2}{\sqrt{u^2 + (v\tilde{\tau})^2}} \left(\frac{\mathbf{H}(\delta\tau)'}{\tau(-\tau^2\delta\mathbf{G} - \mathbf{H})} \mathbf{T} - \frac{\mathbf{G}(\delta\tau)'}{\tau(-\tau^2\delta\mathbf{G} - \mathbf{H})} \mathbf{N} + \frac{\mathbf{B}}{\tau^2} \right), \\ \tilde{\mathbf{B}} &= \frac{v\tilde{\tau}^2(\delta\tau)'}{u(-\tau^2\delta\mathbf{G} - \mathbf{H})} \left(\mathbf{G}\mathbf{T} + \frac{\mathbf{H}}{\tau} \mathbf{N} \right),\end{aligned}$$

where $\mathbf{G}$ and $\mathbf{H}$ are the coefficients,

$$\begin{aligned}\mathbf{G} &= -\delta\tau^2 \left(\delta'' + 2\delta' \frac{\tau'}{\tau} + \delta \frac{\tau''}{\tau} - \delta\tau^2 \right) \\ &\quad - (\delta\tau)' \left(\frac{\kappa}{\tau} - 2\delta'\tau - 3\delta\tau' \right), \\ \mathbf{H} &= (\delta\tau)'(\delta\tau\kappa) - \left(\delta'' + 2\delta' \frac{\tau'}{\tau} + \delta \frac{\tau''}{\tau} - \delta\tau^2 \right).\end{aligned}$$

Proof. Given that γ and $\tilde{\gamma}$ form a $\mathbf{B}$ - $\tilde{\mathbf{O}}$ associated pair, their spatial relationship is defined as:

$$\tilde{\gamma}(s) = \gamma(s) + \delta(s)\mathbf{B}(s). \quad (46)$$

Differentiating Eq. (46) and utilizing the TMOF derivative formulas in Eq. (1) yields:

$$\tilde{\mathbf{T}} = \frac{\mathbf{T} - \delta\tau\mathbf{N} + \frac{(\delta\tau)'}{\tau}\mathbf{B}}{\sqrt{1 + \tau^2(\delta\tau)^2 + ((\delta\tau)')^2}}. \quad (47)$$

Taking the second derivative of Eq. (46) and applying Eq. (1) again, we obtain:

$$\begin{aligned}\tilde{\gamma}'' &= (\delta\tau^2\kappa)\mathbf{T} + \left(\frac{\kappa}{\tau} - 2\delta'\tau - 3\delta\tau' \right) \mathbf{N} \\ &\quad + \left(\delta'' + 2\delta' \frac{\tau'}{\tau} + \delta \frac{\tau''}{\tau} - \delta\tau^2 \right) \mathbf{B}.\end{aligned} \quad (48)$$

The cross product of Eq. (47) and Eq. (48) is:

$$\tilde{\gamma}' \times \tilde{\gamma}'' = \tau\mathbf{G}\mathbf{T} + \mathbf{H}\mathbf{N} + \mathbf{I}\mathbf{B}, \quad (49)$$

where $\mathbf{G}$, $\mathbf{H}$, and $\mathbf{I}$ are explicitly defined as:

$$\begin{aligned}
\mathbf{G} &= -\delta\tau^2 \left(\delta'' + 2\delta' \frac{\tau'}{\tau} + \delta \frac{\tau''}{\tau} - \delta\tau^2 \right) \\
&\quad - (\delta\tau)' \left(\frac{\kappa}{\tau} - 2\delta'\tau - 3\delta\tau' \right), \\
\mathbf{H} &= (\delta\tau)'(\delta\tau\kappa) - \left(\delta'' + 2\delta' \frac{\tau'}{\tau} + \delta \frac{\tau''}{\tau} - \delta\tau^2 \right), \\
\mathbf{I} &= \left(\frac{\kappa}{\tau} - 2\delta'\tau - 3\delta\tau' \right) + \delta^2\tau^3\kappa.
\end{aligned}$$

The norm of the cross product is:

$$\|\tilde{\gamma}' \times \tilde{\gamma}''\| = \tau\sqrt{\mathbf{G}^2 + \mathbf{H}^2 + \mathbf{I}^2}. \quad (50)$$

Using Eq. (50) and Eq. (2) the fields $\tilde{\mathbf{N}}$ and $\tilde{\mathbf{B}}$ are:

$$\begin{aligned}
\tilde{\mathbf{B}} &= \frac{\tilde{\tau}(\tau\mathbf{G}\mathbf{T} + \mathbf{H}\mathbf{N} + \mathbf{I}\mathbf{B})}{\tau\sqrt{\mathbf{G}^2 + \mathbf{H}^2 + \mathbf{I}^2}}, \\
\tilde{\mathbf{N}} &= \left(\frac{(\delta\tau)'\tilde{\tau}\mathbf{H} + \delta\tau^2\tilde{\tau}\mathbf{I}}{\|\tilde{\gamma}'\|\sqrt{\mathbf{G}^2 + \mathbf{H}^2 + \mathbf{I}^2}} \right) \mathbf{T} + \left(\frac{-(\delta\tau)'\tilde{\tau}\mathbf{G} + \tilde{\tau}\mathbf{I}}{\tau\|\tilde{\gamma}'\|\sqrt{\mathbf{G}^2 + \mathbf{H}^2 + \mathbf{I}^2}} \right) \mathbf{N} \\
&\quad + \left(\frac{-\delta\tau^2\tilde{\tau}\mathbf{G} - \tilde{\tau}\mathbf{H}}{\tau\|\tilde{\gamma}'\|\sqrt{\mathbf{G}^2 + \mathbf{H}^2 + \mathbf{I}^2}} \right) \mathbf{B}.
\end{aligned} \quad (51)$$

Evaluating the inner products $\langle \mathbf{B}, \tilde{\mathbf{T}} \rangle = \langle \tilde{\mathbf{O}}, \tilde{\mathbf{T}} \rangle$, $\langle \mathbf{B}, \tilde{\mathbf{N}} \rangle = \langle \tilde{\mathbf{O}}, \tilde{\mathbf{N}} \rangle$ and $\langle \mathbf{B}, \tilde{\mathbf{B}} \rangle = 0$ gives:

$$\begin{aligned}
\frac{\tau(\delta\tau)'}{\sqrt{1 + \tau^2(\delta\tau)^2 + ((\delta\tau)')^2}} &= \frac{u}{\sqrt{u^2 + (v\tilde{\tau})^2}}, \\
\frac{\tau(-\tau^2\delta\mathbf{G} - \mathbf{H})}{\sqrt{1 + \tau^2(\delta\tau)^2 + ((\delta\tau)')^2}\sqrt{\mathbf{G}^2 + \mathbf{H}^2 + \mathbf{I}^2}} &= \frac{v\tilde{\tau}}{\sqrt{u^2 + (v\tilde{\tau})^2}}, \\
\frac{\tau\tilde{\tau}\mathbf{I}}{\sqrt{\mathbf{G}^2 + \mathbf{H}^2 + \mathbf{I}^2}} &= 0.
\end{aligned} \quad (52)$$

Since $\tau, \tilde{\tau} \neq 0$, it follows from Eq. (52) that $\mathbf{I} = 0$. Substituting back into Eq. (47) and Eq. (51) completes the proof. $\square$

Corollary 5. For a base curve γ with non-zero constant torsion ($\tau' = 0$), the geometric condition $\mathbf{I} = 0$ in Eq. (52) reduces to:

$$\frac{\kappa}{\tau} - 2\delta'\tau + \delta^2\tau^3\kappa = 0, \quad (53)$$

which yields the Riccati differential equation:

$$\delta' = \frac{\kappa}{2\tau^2} + \frac{\kappa\tau^2}{2}\delta^2. \quad (54)$$

Using a particular solution δ_1 and the transformation $\delta = \delta_1 + \frac{1}{v}$, where $\delta' = \delta'_1 - \frac{v'}{v^2}$, Eq. (54) becomes:

$$\delta'_1 - \frac{v'}{v^2} = \frac{\kappa}{2\tau^2} + \frac{\kappa\tau^2}{2} \left(\delta_1^2 + \frac{2\delta_1}{v} + \frac{1}{v^2} \right).$$

Since δ_1 satisfies Eq. (54), the relation simplifies to the first-order linear differential equation:

$$v' + (\kappa\tau^2\delta_1)v = -\frac{\kappa\tau^2}{2}. \quad (55)$$

The analytical solution for $v(s)$ using the integrating factor $e^{\int \kappa\tau^2\delta_1 ds}$ is:

$$v(s) = e^{-\int \kappa\tau^2\delta_1 ds} \left(-\int \frac{\kappa\tau^2}{2} e^{\int \kappa\tau^2\delta_1 ds} ds + c \right), \quad (56)$$

where c is a constant. Thus, the general distance function $\delta(s)$ is explicitly obtained as:

$$\delta(s) = \delta_1(s) + \frac{e^{\int \kappa\tau^2\delta_1 ds}}{-\int \frac{\kappa\tau^2}{2} e^{\int \kappa\tau^2\delta_1 ds} ds + c}. \quad (57)$$

Theorem 15. If $\tilde{\gamma}$ denotes the $\mathbf{B}-\tilde{\mathbf{O}}$ associated curve of γ in the TMOF setting, then the curvature $\tilde{\kappa}$ and torsion $\tilde{\tau}$ are given by:

$$\tilde{\kappa} = \frac{u^4(-\tau^2\delta\mathbf{G}-\mathbf{H})}{v(\delta\tau)'(u^2+(v\tilde{\tau})^2)^{\frac{3}{2}}}, \quad \tilde{\tau} = \sqrt[3]{\frac{\Omega_5}{\left(\frac{u(-\tau^2\delta\mathbf{G}-\mathbf{H})}{v(\delta\tau)'}\right)^2}}, \quad (58)$$

where

$$\begin{aligned} \Omega_5 = & \tau\mathbf{G}(3\delta'\kappa\tau^2 + 5\delta\kappa\tau\tau' + \delta\kappa'\tau^2 - \kappa^2) \\ & + \tau^2\mathbf{H}\left(\delta\kappa^2\tau + \frac{\kappa'}{\tau} - 3\delta''\tau - 9\delta'\tau' - 4\delta\tau'' - 3\delta\frac{(\tau')^2}{\tau} + \delta\tau^3\right). \end{aligned}$$

Proof. Differentiating Eq. (46) three times, we obtain:

$$\begin{aligned} \tilde{\gamma}''' = & (3\delta'\kappa\tau^2 + 5\delta\kappa\tau\tau' + \delta\kappa'\tau^2 - \kappa^2)\mathbf{T} \\ & + \left(\delta\kappa^2\tau + \frac{\kappa'}{\tau} - 3\delta''\tau - 9\delta'\tau' - 4\delta\tau'' - 3\delta\frac{(\tau')^2}{\tau} + \delta\tau^3\right)\mathbf{N} \\ & + \left(\kappa - 3\delta'\tau^2 - 6\delta\tau\tau' + \delta''' + 3\delta''\frac{\tau'}{\tau} + 3\delta'\frac{\tau''}{\tau} + \delta\frac{\tau'''}{\tau}\right)\mathbf{B}. \end{aligned} \quad (59)$$

Substituting Eq. (52) and Eq. (59) into the standard curvature and torsion formulas completes the proof. $\square$

Definition 8. Let γ and $\tilde{\gamma}$ be regular space curves defined with the Torsion-Modified Orthogonal Frame (TMOF). If the binormal vector $\mathbf{B}(s)$ of the curve γ and the normal-plane vector $\tilde{\mathbf{P}}(s)$ of the curve $\tilde{\gamma}$ are collinear, i.e., $\mathbf{B}(s) = \tilde{\mathbf{P}}(s)$, then $\tilde{\gamma}$ is called a $\mathbf{B}$ - $\tilde{\mathbf{P}}$ associated curve of γ .

Theorem 16. If $\tilde{\gamma}$ is the $\mathbf{B}$ - $\tilde{\mathbf{P}}$ associated curve of γ in the TMOF setting, then the relationship between their modified orthogonal frames is given by:

$$\begin{aligned}\tilde{\mathbf{T}} &= \frac{1}{\sqrt{1 + \tau^2(\delta\tau)^2}} \mathbf{T} - \frac{\delta\tau}{\sqrt{1 + \tau^2(\delta\tau)^2}} \mathbf{N}, \\ \tilde{\mathbf{N}} &= \frac{\tilde{\tau}}{\sqrt{w^2 + x^2}} \left(\frac{\delta\tau x}{\sqrt{1 + \tau^2(\delta\tau)^2}} \mathbf{T} + \frac{x}{\tau^2 \sqrt{1 + \tau^2(\delta\tau)^2}} \mathbf{N} + \frac{w}{\tau^2} \mathbf{B} \right), \\ \tilde{\mathbf{B}} &= \frac{\tilde{\tau}}{\sqrt{w^2 + x^2}} \left(\frac{-w\delta\tau}{\sqrt{1 + \tau^2(\delta\tau)^2}} \mathbf{T} - \frac{w}{\tau^2 \sqrt{1 + \tau^2(\delta\tau)^2}} \mathbf{N} + \frac{x}{\tau^2} \mathbf{B} \right)..\end{aligned}$$

Proof. The $\mathbf{B}$ - $\tilde{\mathbf{P}}$ associated curve definition imposes the following inner product relations:

$$\langle \mathbf{B}, \tilde{\mathbf{N}} \rangle = \langle \tilde{\mathbf{P}}, \tilde{\mathbf{N}} \rangle, \quad \langle \mathbf{B}, \tilde{\mathbf{B}} \rangle = \langle \tilde{\mathbf{P}}, \tilde{\mathbf{B}} \rangle, \quad \langle \mathbf{B}, \tilde{\mathbf{T}} \rangle = \langle \tilde{\mathbf{P}}, \tilde{\mathbf{T}} \rangle = 0.$$

Evaluating these with respect to the TMOF components yields the system:

$$\begin{aligned}\frac{w}{\sqrt{w^2 + x^2}} &= \frac{\tau(-\tau^2\delta\mathbf{G} - \mathbf{H})}{\sqrt{1 + \tau^2(\delta\tau)^2 + ((\delta\tau)')^2} \sqrt{\mathbf{G}^2 + \mathbf{H}^2 + \mathbf{I}^2}}, \\ \frac{x}{\sqrt{w^2 + x^2}} &= \frac{\tau\mathbf{I}}{\sqrt{\mathbf{G}^2 + \mathbf{H}^2 + \mathbf{I}^2}}, \\ \frac{\tau(\delta\tau)'}{\sqrt{1 + \tau^2(\delta\tau)^2 + ((\delta\tau)')^2}} &= 0.\end{aligned}\tag{60}$$

Since $\tau \neq 0$, Eq. (60) implies $(\delta\tau)' = 0$. Substituting this constraint into the frame formulas yields the expressions in the theorem. $\square$

Corollary 6. In the Torsion-Modified Orthogonal Frame (TMOF) setting, if the binormal vector $\mathbf{B}(s)$ of the base curve γ is collinear with a unit vector in the normal plane $\tilde{\mathbf{P}}(s)$ of its partner curve $\tilde{\gamma}$ (i.e., $\tilde{\gamma}$ is a $\mathbf{B}$ - $\tilde{\mathbf{P}}$ associated curve), then the distance function between the two curves is given by $\delta(s) = \frac{c}{\tau(s)}$, ($c \in \mathbb{R}$). In particular, when the torsion τ of the base curve is constant, the spatial distance δ between the curves remains strictly constant.

Theorem 17. If $\tilde{\gamma}$ is a $\mathbf{B}-\tilde{\mathbf{P}}$ associated curve of γ in the TMOF setting, its curvature $\tilde{\kappa}$ and torsion $\tilde{\tau}$ are given by

$$\tilde{\kappa} = \frac{w^3 \sqrt{w^2 + x^2}}{\delta^3 \tau^8 (1 + \delta^2 \tau^4)^3}, \quad \tilde{\tau} = \frac{\Omega_{13}}{w^2 + x^2}, \quad (61)$$

where Ω_{13} is explicitly given by

$$\begin{aligned} \Omega_{13} &= \delta^2 \tau^5 (2\delta \kappa \tau \tau' + \delta \kappa' \tau^2 - \kappa^2) \\ &\quad + \delta \tau^4 \left(\delta \kappa^2 \tau + \frac{\kappa'}{\tau} - \delta \tau'' + \delta \tau^3 \right) \\ &\quad + \tau^2 \left(\frac{\kappa}{\tau} - \delta \tau' + \delta^2 \tau^3 \kappa \right) (\kappa - 3\delta \tau \tau'). \end{aligned}$$

Proof. Substituting $(\delta \tau)' = 0$ from Eq. (60) and the reduced TMOF components into the standard curvature and torsion formulas yields the result.

□

Definition 9. Let γ and $\tilde{\gamma}$ be regular space curves defined with the Torsion-Modified Orthogonal Frame (TMOF). If the binormal vector $\mathbf{B}(s)$ of the curve γ and the rectifying-plane vector $\tilde{\mathbf{R}}(s)$ of the curve $\tilde{\gamma}$ are collinear, i.e., $\mathbf{B}(s) = \tilde{\mathbf{R}}(s)$, then $\tilde{\gamma}$ is called a $\mathbf{B}-\tilde{\mathbf{R}}$ associated curve of γ .

Theorem 18. If $\tilde{\gamma}$ is the $\mathbf{B}-\tilde{\mathbf{R}}$ associated curve of γ in the TMOF setting, then the relationship between their modified orthogonal frames is given by:

$$\begin{aligned} \tilde{\mathbf{T}} &= \frac{y}{\sqrt{y^2 + (z\tilde{\tau})^2}} \left(\frac{1}{\tau(\delta\tau)'} \mathbf{T} - \frac{\delta\tau}{\tau(\delta\tau)'} \mathbf{N} + \frac{1}{\tau^2} \mathbf{B} \right), \\ \tilde{\mathbf{N}} &= \frac{yz\tilde{\tau}^2}{y^2 + (z\tilde{\tau})^2} \left(\left(\frac{\mathbf{H}}{\tau^2 \mathbf{I}} + \frac{\delta}{(\delta\tau)'} \right) \mathbf{T} + \left(\frac{1}{\tau^3 (\delta\tau)'} - \frac{\mathbf{G}}{\tau^3 \mathbf{I}} \right) \mathbf{N} \right), \\ \tilde{\mathbf{B}} &= \frac{z\tilde{\tau}^2}{\sqrt{y^2 + (z\tilde{\tau})^2}} \left(\frac{\mathbf{G}}{\tau \mathbf{I}} \mathbf{T} + \frac{\mathbf{H}}{\tau^2 \mathbf{I}} \mathbf{N} + \frac{1}{\tau^2} \mathbf{B} \right). \end{aligned}$$

Proof. The $\mathbf{B}-\tilde{\mathbf{R}}$ associated curve definition imposes the following inner product relations:

$$\langle \mathbf{B}, \tilde{\mathbf{T}} \rangle = \langle \tilde{\mathbf{R}}, \tilde{\mathbf{T}} \rangle, \quad \langle \mathbf{B}, \tilde{\mathbf{B}} \rangle = \langle \tilde{\mathbf{R}}, \tilde{\mathbf{B}} \rangle, \quad \langle \mathbf{B}, \tilde{\mathbf{N}} \rangle = \langle \tilde{\mathbf{R}}, \tilde{\mathbf{N}} \rangle = 0.$$

Evaluating these with respect to the TMOF components yields:

$$\begin{aligned} \frac{y}{\sqrt{y^2 + (z\tilde{\tau})^2}} &= \frac{\tau(\delta\tau)'}{\|\tilde{\gamma}'\|}, \\ \frac{z\tilde{\tau}}{\sqrt{y^2 + (z\tilde{\tau})^2}} &= \frac{\tau \mathbf{I}}{\sqrt{\mathbf{G}^2 + \mathbf{H}^2 + \mathbf{I}^2}}, \\ \frac{\tau \tilde{\tau} (-\tau^2 \delta \mathbf{G} - \mathbf{H})}{\|\tilde{\gamma}'\| \sqrt{\mathbf{G}^2 + \mathbf{H}^2 + \mathbf{I}^2}} &= 0. \end{aligned} \quad (62)$$

Substituting Eq. (62) into Eqs. (47) and (51) completes the proof. $\square$

Theorem 19. If $\tilde{\gamma}$ is the $\mathbf{B}$ - $\tilde{\mathbf{R}}$ associated curve of γ in the TMOF setting, its curvature $\tilde{\kappa}$ and torsion $\tilde{\tau}$ are given by:

$$\begin{aligned}\tilde{\kappa} &= \frac{\tau \mathbf{I} y^3}{z \tilde{\tau} ((\delta \tau)')^3 (y^2 + (z \tilde{\tau})^2)}, \\ \tilde{\tau} &= \frac{\Omega_7 z^2 \pm \sqrt{\Omega_7^2 z^4 - 4 \tau^4 \mathbf{I}^4 y^2 z^2}}{2 \tau^2 \mathbf{I}^2 z^2},\end{aligned}\quad (63)$$

where the auxiliary function Ω_7 is explicitly defined as:

$$\begin{aligned}\Omega_7 &= \tau \mathbf{G} (3 \delta' \kappa \tau^2 + 5 \delta \kappa \tau \tau' + \delta \kappa' \tau^2 - \kappa^2) \\ &\quad + \tau^2 \mathbf{H} \left(\delta \kappa^2 \tau + \frac{\kappa'}{\tau} - 3 \delta'' \tau - 9 \delta' \tau' - 4 \delta \tau'' - 3 \delta \frac{(\tau')^2}{\tau} + \delta \tau^3 \right) \\ &\quad + \tau^2 \mathbf{I} \left(\kappa - 3 \delta' \tau^2 - 6 \delta \tau \tau' + \delta''' + 3 \delta'' \frac{\tau'}{\tau} + 3 \delta' \frac{\tau''}{\tau} + \delta \frac{\tau'''}{\tau} \right).\end{aligned}\quad (64)$$

Proof. Substituting Eq. (62) and Eq. (59) into the standard curvature and torsion formulas yields $\tilde{\kappa}$ and a quadratic equation for $\tilde{\tau}$. Solving this quadratic analytically gives $\tilde{\tau}$, provided $\Omega_7^2 z^4 - 4 \tau^4 \mathbf{I}^4 y^2 z^2 \geq 0$ for real solutions. $\square$

Corollary 7. Let γ be a curve with non-zero constant curvature and constant torsion. Then, for the connection coefficient function $\delta(s)$ between the curves, the following second-order non-linear differential equation is obtained:

$$\delta'' - \frac{2 \delta \tau^4}{1 + \delta^2 \tau^4} (\delta')^2 - \delta \tau^2 = 0.$$

Proof. When $\kappa = \tau = \text{constant}$, the system coefficients $\mathbf{G}$ and $\mathbf{H}$ reduce to

$$\begin{aligned}\mathbf{G} &= -\delta \delta'' \tau^2 + \delta^2 \tau^4 - \delta' \kappa + 2 (\delta')^2 \tau^2, \\ \mathbf{H} &= \delta \delta' \tau^2 \kappa - \delta'' + \delta \tau^2.\end{aligned}$$

Substituting these reduced expressions into Eq. (62), which is given by

$$\delta \tau \kappa \mathbf{G} + \mathbf{H} = 0, \quad (65)$$

yields the desired second-order non-linear differential equation. $\square$

Examples

In this section, we illustrate the proposed associated curve construction within the torsion modified orthogonal frame using a dynamic mathematical example. To demonstrate the flexibility of our framework on curves with

variable curvature and torsion, we focus on the $\mathbf{T}\text{--}\tilde{\mathbf{P}}$ configuration by utilizing a catenary curve.

Example 1. Consider the catenary curve:

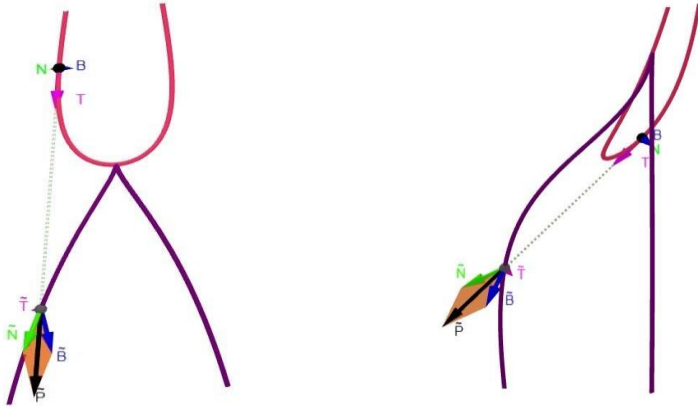
$$\gamma(s) = (\cosh s, \sinh s, \cosh s). \quad (66)$$

The curvature and torsion are non-constant, and the structural coefficients u, v, w, x, y, z are smooth functions of s . For the $\mathbf{T}\text{--}\tilde{\mathbf{P}}$ associated partner, Theorem 2 gives the distance function $\delta(s) = -s + c$. Taking $c = 0$, the partner curve $\tilde{\gamma}(s) = \gamma(s) - s\mathbf{T}(s)$ is:

$$\begin{aligned} \tilde{\gamma}(s) &= (\cosh s, \sinh s, \cosh s) - s(-\sinh s, \cosh s, \sinh s) \\ &= (\cosh s + s\sinh s, \sinh s - s\cosh s, \cosh s - s\sinh s). \end{aligned} \quad (67)$$

Figure 1 validates the involute relationship under the TMOF, visually confirming that the tangent vector $\mathbf{T}$ of γ lies within the normal plane $\tilde{\mathbf{P}}$ of $\tilde{\gamma}$ at every point.

While the $\mathbf{T}\text{--}\tilde{\mathbf{P}}$ configuration is explicitly visualized, Theorem 1 precludes the existence of $\mathbf{T}\text{--}\tilde{\mathbf{O}}$ for $\tau \neq 0$. Additionally, the $\mathbf{T}\text{--}\tilde{\mathbf{R}}$ configuration yields a complex integral without a standard closed-form solution. Thus, the $\mathbf{T}\text{--}\tilde{\mathbf{P}}$ family demonstrates the geometric validity of TMOF transitions.



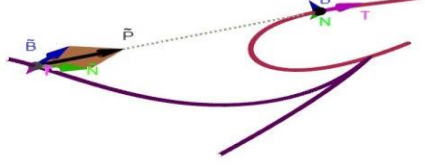


Figure 1: A family of $\mathbf{T}-\tilde{\mathbf{P}}$ associated partner curves generated for integration constants $c = 0$, $c = 2$, and $c = -2$. The pink curve represents the spatial base curve γ (catenary), while the purple curve manifests its corresponding associated partner $\tilde{\gamma}$.

CONCLUSION

In this paper, we established a comprehensive mathematical framework for characterizing associated curves in Euclidean 3-space by employing the torsion modified orthogonal frame. Our systematic investigation of the tangent, principal normal, and binormal associated configurations has yielded the following primary geometric outcomes:

- We proved that a regular spatial curve with non-zero torsion cannot admit a $\mathbf{T}-\tilde{\mathbf{O}}$ associated partner, as the structural alignment forces the torsion to vanish identically.
- The $\mathbf{T}-\tilde{\mathbf{P}}$ configuration recovers the family of generalized planar involutes. Our analysis demonstrates that varying the integration constant generates a family of parallel planar involutes.
- Under the assumption of constant torsion, the $\mathbf{N}-\tilde{\mathbf{P}}$ and $\mathbf{B}-\tilde{\mathbf{P}}$ associations naturally generate partner curves with constant distance functions, exhibiting geometric behaviors analogous to classical Bertrand and Mannheim pairs.

The explicit formulations for curvature, torsion, and distance constraints obtained in this work provide a stable foundation for potential

applications in geometric modeling and kinematic design. For future studies, this continuous frame approach can be extended to other spaces and alternative frames.

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